EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH2581-WE01

Title:
Algebra II

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Let

$$R = \{\frac{a}{5^r 7^s} \in \mathbb{Q} \mid a \in \mathbb{Z},\ r, s \in \mathbb{Z}_{\geq 0}\}.$$

1.1 Show that R, with binary operations inherited from $\mathbb{Q}$, constitutes a ring.

1.2 Is R an integral domain? (Provide your reasoning!)

1.3 Consider the subset

$$S = \{\frac{3a}{5^r 7^s} \in \mathbb{Q} \mid a \in \mathbb{Z},\ r, s \in \mathbb{Z}_{\geq 0}\} \subset R.$$

Show that S is a proper ideal of R (i.e. an ideal that is neither $\{0_R\}$ nor R), and determine whether S is maximal or not.

1.4 Show that, for any $n \geq 1$, there is a polynomial of degree n which is a unit in the ring $(\mathbb{Z}/25)[x]$.

1.5 Determine all elements in $\mathbb{Z}[\sqrt{-13}]$ that divide 14.

1.6 Denote by $d_1 d_2$ the two digit integer at the end of your (6 characters long) anonymous code, and let $r \in \{0, 1, 2\}$ be the remainder of the division of $d_1 d_2$ by 3.
Decompose the polynomial

$$x^4 + (2 + 4r)x^2 + 4rx + (3 + 4(r - 4r^2)) \text{ in } (\mathbb{Z}/11)[x]$$

into irreducible factors. (Justify irreducibility for each factor.)

Q2 Let $d \in \mathbb{Z}_{>0}$ and $S = \mathbb{Z}[\sqrt{-d}]$, and let p be an odd prime.

2.1 For $n \in \mathbb{Z}$ define the maps $\varphi_n : S \to \mathbb{Z}/p$, given for a, $b \in \mathbb{Z}$ by

$$\varphi_n(a + b\sqrt{-d}) = a^p + nb^p \pmod{p}.$$

Characterise those $n \in \mathbb{Z}$ for which φ_n is a ring homomorphism, in terms of d and p.
(Justify your claim.)

2.2 Let $R = (\mathbb{Z}/13)[x]$.
Is the ideal $I = (x^2 + \overline{2}x - \overline{3}, x^3 + \overline{7})_R$ a principal ideal?

For the remainder of this question, let $R = \mathbb{Z}[\sqrt{-5}]$ and consider $I = (2)_R$.

2.3 Determine how many cosets there are in R with respect to I, and write down a representative of each of these cosets.

2.4 For each of the following statements either prove it or give a reason why it is not correct:

(i) R/I is a commutative ring.

(ii) Every non-zero element in R/I has an inverse.

(iii) There are no zero divisors in R/I.

(iv) R/I is isomorphic to a product of two non-trivial rings.

Q3 Let $I = (4+3\sqrt{7})_R$ and $J = (5-4\sqrt{7})_R$ be ideals in $R = \mathbb{Z}[\sqrt{7}]$.

3.1 Show that $I + J = \mathbb{Z}[\sqrt{7}]$.

3.2 Determine whether the following statement is correct or incorrect:
The quotient ring $\mathbb{Z}[\sqrt{7}]/(64+\sqrt{7})_R$ is isomorphic to the direct product of the rings $\mathbb{Z}[\sqrt{7}]/I$ and $\mathbb{Z}[\sqrt{7}]/J$.
(Carefully cite any result that you use.)

3.3 Suppose the group G acts on a finite non-empty set X.
For each of the following two statements give a proof or a counterexample.

(i) The stabilisers of two elements in X in the same orbit under G have the same size.

(ii) The orbits under G of any two elements in X have the same size.

3.4 Write down all the subsets of $S_4 \times \mathbb{Z}/2$, up to swapping letters in S_4, that constitute a subgroup which is isomorphic to the Klein 4-group.
Moreover, choose one of these subgroups, say H, and determine whether each left coset with respect to H is also a right coset.

Q4 **4.1** For each of the following pairs of groups, either prove that they are isomorphic or provide a reason why they are not.

$$\text{a) } D_3 \times D_3 \text{ and } A_4 \times \mathbb{Z}/3, \quad \text{b) } (\mathbb{Z}/24)^\times \text{ and } \mathbb{Z}/2 \times \mathbb{Z}/4.$$

4.2 Which of the following permutations σ_i, if any, are conjugate in S_6?

$$\sigma_1 = (3\,4\,5\,1)(1\,6\,5)(1\,2),\ \sigma_2 = (1\,2)(1\,6\,5)(3\,4\,5\,1),\ \sigma_3 = (2\,3\,5)(1\,4\,6)\,.$$

Moreover, write σ_1 and σ_2 as a product of 3-cycles or show that it is not possible to do so.
Finally, if σ_i and σ_j are conjugate in S_6 for $i \neq j$ then find a $g \in S_6$ such that $\sigma_i = g\sigma_j g^{-1}$.

4.3 Classify all the abelian groups of order 2025 and determine the number of elements of precise order 3 for each.

4.4 Let p be an odd prime. Write down all the possible groups, up to isomorphism, that can occur as *proper* subgroups of groups of order $2p^2$.

Q5 **5.1** Show that the centre $Z(G)$ of a group G is a subgroup of G. Also show that it is normal in G.

5.2 Determine the centre of $\mathrm{GL}_2(\mathbb{Z}/5)$.

5.3 Show that the order of any conjugacy class in G divides $\#G$.
(Carefully formulate any results that you use.)

5.4 Let G be a *non-abelian* group of order 16.
Determine all the possible sizes that the centre of G can have, and give an example for each of these cases.

5.5 Let $n \geq 5$, and let H be a normal subgroup of A_n which contains a 5-cycle.
For each of the following two statements give a proof or a counterexample:

(i) Under the above assumptions H contains a 3-cycle.

(ii) Under the above assumptions, assuming also that H contains a 3-cycle, then H must agree with A_n.

Carefully cite any results from lectures that you use.