EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH2627-WE01

Title:
Geometric Topology II

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

In all of the following, you are allowed to use any result from the lectures notes provided you state it correctly.

Q1 We think of S^1 as being the unit complex numbers $\{z \in \mathbb{C} : |z| = 1\}$. Given a function $f : D \to \mathbb{C}$, where D is some subset of $\mathbb{C}$, we set

$$\gamma_f : S^1 \to S^1, \qquad \gamma_f(z) = \frac{f(z)}{|f(z)|}$$

if this is well-defined and continuous. In the following, we denote by $\boldsymbol{\omega}(\gamma_f)$ the winding number of such γ_f. Further, for $c \in \mathbb{C}$, we may write $f(c\,\cdot)$ to denote the function given by

$$z \mapsto f(cz).$$

1.1 Determine for which of the following functions f the map γ_f is well-defined and continuous and determine $\boldsymbol{\omega}(\gamma_f)$ in those cases.

(i) $f(z) = z^5 - 1/2$

(ii) $f(z) = \frac{z^4-4}{z^3+1/3}$

(iii) $f(z) = \frac{5z^2+1/3}{3z^2-1/3}$

(iv) $f(z) = z|z|^5 - 2z^3 + 1/2$

1.2 Find two polynomials p and q of degree 7 such that

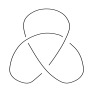
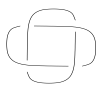
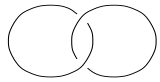
$$\boldsymbol{\omega}(\gamma_p) = \boldsymbol{\omega}(\gamma_q) = 3$$

and

$$\boldsymbol{\omega}(\gamma_{p(2\,\cdot)}) = 4 \quad \text{and} \quad \boldsymbol{\omega}(\gamma_{q(2\,\cdot)}) = 3.$$

1.3 Given any two polynomials p and q which satisfy the conditions specified in **1.2**, is there some $r > 2$ such that $\boldsymbol{\omega}(\gamma_{p(r\,\cdot)}) = \boldsymbol{\omega}(\gamma_{q(r\,\cdot)})$? Justify your answer.

Q2 In the following, you may find helpful that the figure-8 knot, the left-hand trefoil and the right-hand trefoil are the only non-trivial knots with crossing number not bigger than 4. Moreover, any of the following relations from the lectures and problem sheets may be useful.

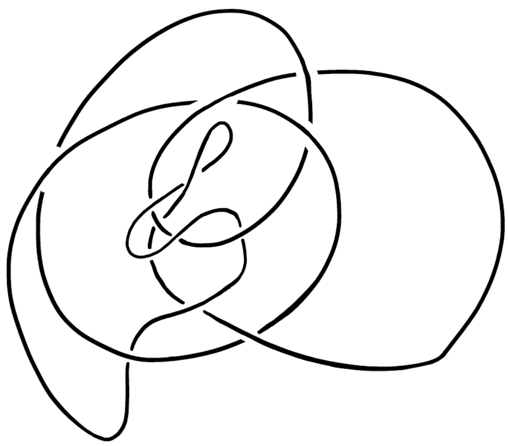
$T =$  $S =$  $H =$ 

$$V_T(t) = -t^4 + t^3 + t$$
$$\langle H \rangle = -(A^4 + A^{-4})$$
$$\langle S \rangle = -A^{10} + A^6 - A^2 - A^{-6}$$

$\Big\langle$  $\Big\rangle = -A^3 \Big\langle$  $\Big\rangle$ $\qquad$ $\Big\langle$  $\Big\rangle = -A^{-3} \Big\langle$  $\Big\rangle$

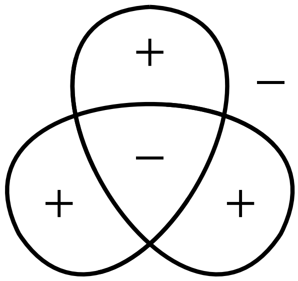
2.1 Transform the below diagram into a reduced alternating diagram. Only use Reidemeister moves and clearly specify where which move takes place. Is your new diagram regular isotopic to the original one? Justify your answer.

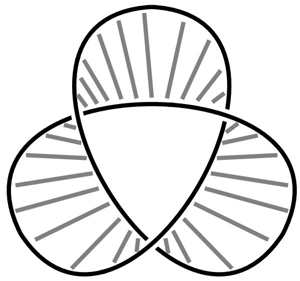
2.2 Compute the Jones polynomial of the associated knot K.



2.3 Is K achiral? Can K be the composition of two non-trivial alternating knots? Justify your answers.

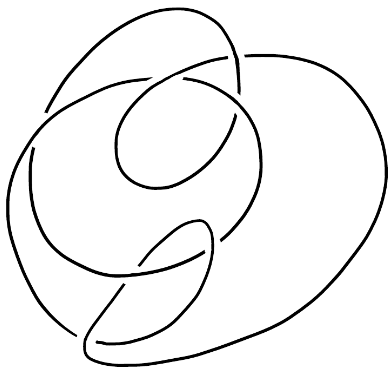
Q3 Given a knot diagram D, recall that D cuts the plane into several regions (one of which is unbounded). It is a fact (which you can take for granted) that we can always label these regions with labels + and − such that adjacent regions have distinct labels, see the left figure below.





Assuming that D is reduced, we build a compact connected surface Z_D as follows. First, we label the regions in the above fashion such that the unique unbounded region carries the label −. Now, each of the + labelled regions forms a disk. Two such disks may intersect which happens exactly at the crossings of the original diagram D. We replace these intersections by twists taking care of over- and underpasses similarly as in Seifert's algorithm. The figure on the right shows a sketch of the surface Z_D when D is the standard diagram of the left-hand trefoil.

3.1 Let D be the following diagram. Label D as described and identify Z_D, that is, determine the values of some topological invariants which specify Z_D up to equivalence.



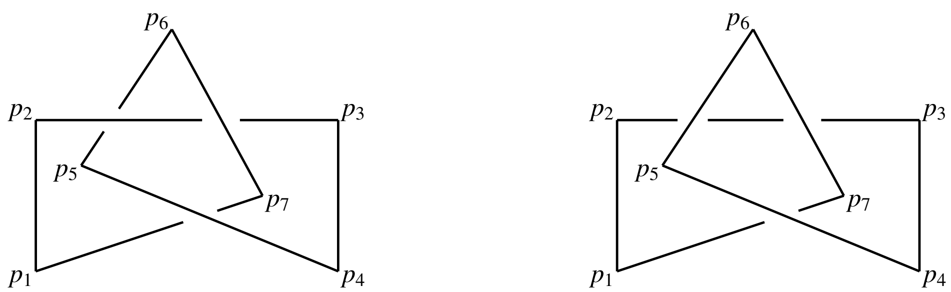
3.2 Identify the surface S_D obtained by applying Seifert's algorithm to the diagram D. Are S_D and Z_D equivalent? Justify your answer.

3.3 Show that there is a sequence of diagrams D_n such that Z_{D_n} is equivalent to S_{D_n} and $c(K_n) \to \infty$, where K_n denotes the knot represented by D_n and $c(K_n)$ is the corresponding crossing number. *[Hint: Search for D_n such that Z_{D_n} is orientable.]*

Q4 In this problem, we take a look at knot diagrams where each arc is comprised of straight line segments. Given such a diagram, we say it is **spanned by n points** if it can be constructed in the following fashion.

- First, pick n distinct points $p_1, p_2, \dots, p_n$ in the plane.
- Second, connect them cyclically through line segments. That is, draw a line segment between consecutive points (i.e., between p_1 and p_2, between p_2 and p_3 etc.) and a line segment from p_n to p_1.
- Finally, consider the line segments from the previous step. Suppose that at each crossing of such line segments exactly two (and no more) distinct line segments meet. If this is the case, turn the interior intersections of the line segments (i.e., those intersections different from any of the points $p_1, p_2, \dots, p_n$) into crossings of a knot diagram by choosing one of the two intersecting segments to provide the underpasses and the other one to provide the overpass.

See the below figure for a diagram of the trefoil knot (left) and the unknot (right) both of which are spanned by (the same) 7 points.



Every knot can be represented by a diagram which is spanned by finitely many points. We define the **spanning number** of a knot K to be

$$s(K) = \min\{n : \text{ there is a diagram for } K \text{ which is spanned by } n \text{ points}\}.$$

In all of the following, justify your answers.

4.1 Is there a knot with spanning number equal to 4?

4.2 What is the spanning number of the figure-8 knot?

4.3 We denote the crossing number of a knot K by $c(K)$. Determine the largest integer b such that the following relation holds for each knot K.

$$c(K) \leq 1/2 \cdot (s(K) - b) \cdot s(K)$$