Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH2647-WE01

Title:

Probability II

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 A random symbol generator produces each of 10 digits and each of 26 letters of the latin alphabet with positive probability. Assuming that the individual outcomes are independent, let M be event that an infinite sequence of such symbols contains the word MATH2647 infinitely often.

1.1 Find the probability of M using a suitable monotone approximation.

1.2 Find the probability of M using the Borel-Cantelli lemma.

In your answer you should clearly state and carefully apply every result you use.

Q2 An XOR gate adds bits according to the following rules:

$$0 + 0 = 1 + 1 = 0\,, \qquad 1 + 0 = 0 + 1 = 1\,.$$

Suppose that random bits $(B_k)_{k=1}^n$ are independent and have a common distribution $\mathsf{P}(B_k = 1) = p = 1 - \mathsf{P}(B_k = 0)$, where $0 < p < 1$. Write S_n for the result of the sum of these bits using XOR gates, and let p_n be the probability of the event $\{S_n = 1\}$, equivalently,

$$p_n = \mathsf{P}\Big(\text{the sequence } (B_k)_{k=1}^n \text{ contains an odd number of `ones'}\Big)\,.$$

2.1 Compute p_1 and p_2.

2.2 Show that, with properly defined p_0, we have $p_n = p + (1 - 2p)\,p_{n-1}$ for all $n \geq 1$.

2.3 Use generating functions to derive a closed formula for p_n, and check that it gives correct values for $n = 0, 1, 2$.

2.4 Find $\lim\limits_{n\to\infty} p_n$ and explain your result.

Q3 Consider a Markov chain with state space $\{1, 2, 3, 4\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

3.1 Describe the class structure of this Markov chain and determine the period of each state.

3.2 Find all stationary distributions for this Markov chain.

3.3 Find a closed expression in terms of powers of the eigenvalues of $\mathbf{P}$ for the n-step transition probabilities $p_{11}^{(n)}$ and $p_{12}^{(n)}$.

3.4 Deduce closed expressions in terms of powers of the eigenvalues of $\mathbf{P}$ for the n-step transition probabilities $p_{13}^{(n)}$ and $p_{14}^{(n)}$, where $n > 0$.

3.5 Classify all states of this Markov chain into transient and recurrent.

In your answer you should clearly state and carefully apply every result you use.

Q4 For real $a_n > 0$ and $p_n \in (0, 1)$, let $(X_n)_{n \geq 1}$ be random variables such that

$$P(X_n = a_n) = p_n = 1 - P(X_n = 0).$$

For each of the following claims, prove the result if it is correct, or find a counterexample otherwise:

4.1 If $a_n \to 0$, then $X_n \to 0$ in L^r, for some $r > 0$, as $n \to \infty$.

4.2 If $a_n \to 0$, then $X_n \to 0$ in probability as $n \to \infty$.

4.3 If $a_n \to 0$, then $X_n \to 0$ almost surely as $n \to \infty$.

4.4 If $p_n \to 0$, then $X_n \to 0$ in L^r, for some $r > 0$, as $n \to \infty$.

4.5 If $p_n \to 0$, then $X_n \to 0$ in probability as $n \to \infty$.

4.6 If $p_n \to 0$, then $X_n \to 0$ almost surely as $n \to \infty$.

4.7 If $p_n \to 0$ and the variables X_n are independent, then $X_n \to 0$ in L^r, for some $r > 0$, as $n \to \infty$.

4.8 If $p_n \to 0$ and the variables X_n are independent, then $X_n \to 0$ in probability as $n \to \infty$.

4.9 If $p_n \to 0$ and the variables X_n are independent, then $X_n \to 0$ almost surely as $n \to \infty$.

In your answer you should clearly state and carefully apply any result you use.