Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH2657-WE01

Title:
Special Relativity and Electromagnetism II

Time (for guidance only):	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

Q1 **1.1** Discuss the validity of each of the following claims, including detailed calculations to justify any statements that you make.

- The set of Galilean boosts form a group.
- The set of proper orthochronous Lorentz boosts form a group.

1.2 A cloaked Romulan Warbird is following the Enterprise, matching its heading and sneaking up behind it. Just as the Warbird passes the Enterprise, interference from the warp drive of the Enterprise causes the Warbird to decloak, so that it becomes visible to the Enterprise. Spock is on the bridge of the Enterprise and measures that the Warbird is moving directly away from the Enterprise at a constant speed of one quarter of the speed of light.

Spock records that it takes two seconds from the time that the Warbird becomes visible for the Enterprise to fire a torpedo at the Warbird. Due to routine maintenance, photon torpedoes are unavailable, so a conventional torpedo is used, which only travels at 75% of the speed of light, as measured by Spock. When the torpedo reaches the Warbird the deflector shields of the Warbird cause the torpedo to immediately bounce back, so that it heads towards the Enterprise. The Captain of the Warbird measures that the speed of the torpedo after the deflection is only half the speed that she measured for the torpedo before the deflection.

Calculate the time elapsed, as measured by Spock, between the Warbird becoming visible and the torpedo returning to hit the Enterprise.

Q2 **2.1** In the frame $\mathcal{R}$ there is no electric field but there is a constant magnetic field of magnitude $|\mathbf{B}|$ that points in the positive z direction. In the frame $\mathcal{R}'$, given by applying a Lorentz transformation (with a and b positive constants)

$$L = \begin{bmatrix} b^3 & -ab & -ab^2 & -ab^3 \\ -ab^3 & b & a^2b^2 & a^2b^3 \\ -ab^2 & 0 & b & a^2b^2 \\ -ab & 0 & 0 & b \end{bmatrix},$$

the magnitude of the magnetic field is measured to be four times the value measured in the frame $\mathcal{R}$. Calculate, in terms of $|\mathbf{B}|$ and the speed of light c, the electric field $\mathbf{E}'$ measured in the frame $\mathcal{R}'$.

2.2 In LineWorld there is only one spatial dimension, rather than the usual three. Discuss the transformation of the totally antisymmetric tensor $\varepsilon^{\mu\nu}$ (with sign convention $\varepsilon^{01} = 1$) under a general Lorentz transformation.

Given any contravariant vector V^μ define $W^\mu = \varepsilon^{\mu\nu} V_\nu$.

In a frame $\mathcal{R}$ it is known that $V^\mu = (12\ell, 24\ell)$, where ℓ is an invariant constant. In the frame $\mathcal{R}'$ it is found that $W'^\mu = (-39\ell, 33\ell)$. Find the rapidity of the Lorentz boost that relates the frames $\mathcal{R}$ and $\mathcal{R}'$.

Q3 **3.1** A closed surface S consists of three pieces $S = S_{\text{cone}} \cup S_{\text{disc}} \cup S_{\text{curved}}$, where

$$\begin{aligned} S_{\text{cone}} &= \{3x^2 + 3y^2 = (z - 8a)^2,\ \ 2a \le z \le 8a\}, \\ S_{\text{disc}} &= \{x^2 + y^2 \le 12a^2,\ \ z = -2a\}, \\ S_{\text{curved}} &= \{x^2 + y^2 + z^2 = 16a^2,\ \ |z| \le 2a\} \end{aligned}$$

with constant $a > 0$. Let $\psi_{\text{cone}}, \psi_{\text{disc}}, \psi_{\text{curved}}$ be the electric flux through each of these three pieces of the surface due to an electric charge at the origin.
Determine the ratios $\psi_{\text{disc}}/\psi_{\text{curved}}$ and $\psi_{\text{cone}}/\psi_{\text{disc}}$.

3.2 Let n be a positive integer and q a non-zero constant. A ball of radius R centred at the origin contains a charge density

$$\rho_n = \frac{q}{R^4}(R^3 - 4r^3)\frac{z^n}{r^{n+2}},$$

where $r^2 = x^2 + y^2 + z^2$.
Calculate the total charge Q_n and the dipole moment $\mathbf{p}_n$ of the ball. Determine the values of n for which $|\mathbf{p}_n| \ge |\mathbf{p}_7|$.

3.3 A magnetic vector potential is given by

$$\mathbf{A} = \beta \exp\left(-\frac{\alpha}{2}(x^2 + y^2 + z^2)\right)(x + z - y,\ x + y - z,\ y + z - x),$$

where α and β are positive constants.
Calculate this magnetic vector potential in Coulomb gauge.

Q4 **4.1** Given any electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, define the associated complex vector $\mathbf{W} = \mathbf{E} + ic\mathbf{B}$, where c is the speed of light. Write the source-free Maxwell equations as a pair of equations for $\mathbf{W}$ and show that the ansatz $\mathbf{W} = (\boldsymbol{\nabla} g) \times (\boldsymbol{\nabla} h)$ automatically solves one of these equations, for all complex functions g, h.

4.2 Find an expression, in terms of derivatives of g and h, for a complex vector $\mathbf{M}$ such that $\frac{\partial \mathbf{W}}{\partial t} = \boldsymbol{\nabla} \times \mathbf{M}$, and show that $\mathbf{W} \cdot \mathbf{M} = 0$.

4.3 Find an equation, involving only the derivatives of g and h, that guarantees that the source-free Maxwell equations are solved. Derive two relations for the associated electric and magnetic fields, one concerning the magnitudes of these fields and the other specifying the angle between these fields.

4.4 For the choice $g = ia(z - ct)$, where a is an arbitrary non-zero real constant, show that a time-independent solution for h can be written in terms of an arbitrary function of a single variable. Make a suitable choice for this function to obtain electric and magnetic fields that are constant but non-vanishing. Make a second choice for this function to obtain electric and magnetic fields that are linear in the spatial coordinates. For both choices, confirm that the two relations from the previous part are satisfied.