Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH2667-WE01

Title:
Monte Carlo II

Time (for guidance only):	1 hour 30 minutes	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Let X be a logistic random variable, $X \sim \text{Logistic}(0, 1)$. Its probability density function is given by

$$f_X(x) = \frac{\exp(-x)}{(1 + \exp(-x))^2}, \ x \in \mathbb{R}.$$

1.1 Derive the cumulative distribution function of X.

1.2 Let Y and Z be two independent exponentially distributed random variables, both with mean 1. Show that

$$\ln\left(\frac{Y}{Z}\right) \sim \text{Logistic}(0, 1).$$

1.3 Using the result in **1.2**, derive an algorithm to simulate a sample from the Logistic(0, 1) distribution. You may assume you have access to samples from the Uniform(0, 1) distribution.

1.4 Verify that

$$\frac{Y}{Y + Z} \sim \text{Uniform}(0, 1)$$

and hence deduce that

$$\ln\left(\frac{U}{1 - U}\right) \sim \text{Logistic}(0, 1)$$

where U is uniformly distributed in $(0, 1)$, $U \sim \text{Uniform}(0, 1)$.

1.5 Using the result in **1.4**, outline an algorithm for simulating a sample from the Logistic(0, 1) distribution.

1.6 Show that the method in **1.5** is equivalent to the Inverse Transform Method.

Q2 Let X be a folded Normal variable with probability density function

$$f_X(x) = \begin{cases} \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), & x \geq 0 \\ 0 & x < 0 \end{cases}$$

2.1 Show that the exponential distribution with mean 1 (Exp(1)) can be used as the proposal distribution for X in the acceptance-rejection algorithm.

2.2 Write down the acceptance-rejection algorithm for sampling from X with the Exp(1) distribution as the proposal distribution.

2.3 Now assume that Exp(λ), $\lambda > 0$, is used as a candidate for the proposal distribution. For which values of λ is Exp(λ) a valid proposal?

2.4 If we wish to minimize the number of samples rejected by this algorithm, what would the optimal value of λ be?

2.5 Using the results in **2.1**-**2.4**, derive an algorithm for sampling from the standard Normal distribution, Normal$(0, 1)$.

2.6 Imagine now that $f_X(x)$ is very expensive to compute and we want to reduce the number of evaluations of $f_X(x)$ (this is not actually true in this question but such cases do exist). Suppose that we can find a function g such that $g(x) \leq f_X(x)$ for all $x \geq 0$. Discuss how you can use this finding to reduce the number of evaluations of f_X. Choose a suitable function g to illustrate the method. You will receive more marks for a choice of $g(x)$ that avoids the need to calculate an exponential (or other special functions such as logarithms and trigonometric functions) in steps where $f_X(x)$ will not be evaluated.

Q3 Let U be a Uniform random variable, $U \sim$ Uniform$(0, 1)$.

3.1 Show that

$$E\left((U+1)^{-1}\right) = \ln(2)$$

and calculate

$$\text{Var}\left((U+1)^{-1}\right).$$

Propose a Monte Carlo method for estimating ln(2).

3.2 Let $f(t) = \frac{1}{t+1}$, $t > -1$. Compute

$$\text{Cov}(f(U), f(1-U)).$$

3.3 Set

$$X = f(U) \quad \text{and} \, Y = \frac{1}{2}\left(f(U) + f(1-U)\right).$$

Show that Var$(Y) <$ Var(X).

3.4 Based on the result in **3.3**, write an algorithm to estimate ln(2). You may assume you have access to samples from the Uniform$(0, 1)$ distribution. Compare the algorithm in **3.1** to the one derived in this item.

3.5 Generalize the results in the previous items to estimate $\ln(z)$, $z > 1$.