

## **EXAMINATION PAPER**

Examination Session: May/June

2021

Year:

Exam Code:

MATH2667-WE01

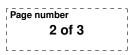
Title:

Monte Carlo II

Time (for guidance only):	1 hour 30 minutes		
Additional Material provided:			
Materials Permitted:			
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.	

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.				
	Please start each question on a new	uestion on a new page.			
	Please write your CIS username at the top of each page.				
	To receive credit, your answers mus explain your reasoning.	t show your	working and		

**Revision:** 



**Q1** Let X be a logistic random variable,  $X \sim \text{Logistic}(0, 1)$ . Its probability density function is given by

$$f_X(x) = rac{\exp(-x)}{\left(1 + \exp(-x)
ight)^2}, \ x \in \mathbb{R}.$$

- **1.1** Derive the cumulative distribution function of *X*.
- **1.2** Let *Y* and *Z* be two independent exponentially distributed random variables, both with mean 1. Show that

$$\ln\left(\frac{Y}{Z}\right) \sim \text{Logistic}(0, 1).$$

- **1.3** Using the result in **1.2**, derive an algorithm to simulate a sample from the Logistic(0, 1) distribution. You may assume you have access to samples from the Uniform(0, 1) distribution.
- 1.4 Verify that

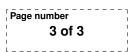
$$\frac{Y}{Y+Z} \sim \text{Uniform}(0,1)$$

and hence deduce that

$$\ln\left(\frac{U}{1-U}\right) \sim \text{Logistic}(0,1)$$

where *U* is uniformly distributed in (0, 1),  $U \sim \text{Uniform}(0, 1)$ .

- **1.5** Using the result in **1.4**, outline an algorithm for simulating a sample from the Logistic(0, 1) distribution.
- **1.6** Show that the method in **1.5** is equivalent to the Inverse Transform Method.



**Q2** Let *X* be a folded Normal variable with probability density function

$$f_X(x) = \begin{cases} rac{2}{\sqrt{2\pi}} \exp\left(-rac{x^2}{2}
ight), & x \ge 0 \\ 0, & x < 0 \end{cases}$$

- **2.1** Show that the exponential distribution with mean 1 (Exp(1)) can be used as the proposal distribution for *X* in the acceptance-rejection algorithm.
- **2.2** Write down the acceptance-rejection algorithm for sampling from X with the Exp(1) distribution as the proposal distribution.
- **2.3** Now assume that  $Exp(\lambda)$ ,  $\lambda > 0$ , is used as a candidate for the proposal distribution. For which values of  $\lambda$  is  $Exp(\lambda)$  a valid proposal?
- **2.4** If we wish to minimize the number of samples rejected by this algorithm, what would the optimal value of  $\lambda$  be?
- **2.5** Using the results in **2.1-2.4**, derive an algorithm for sampling from the standard Normal distribution, Normal(0, 1).
- **2.6** Imagine now that  $f_X(x)$  is very expensive to compute and we want to reduce the number of evaluations of  $f_X(x)$  (this is not actually true in this question but such cases do exist). Suppose that we can find a function g such that  $g(x) \le f_X(x)$  for all  $x \ge 0$ . Discuss how you can use this finding to reduce the number of evaluations of  $f_X$ . Choose a suitable function g to illustrate the method. You will receive more marks for a choice of g(x) that avoids the need to calculate an exponential (or other special functions such as logarithms and trigonometric functions) in steps where  $f_X(x)$  will not be evaluated.
- **Q3** Let *U* be a Uniform random variable,  $U \sim \text{Uniform}(0, 1)$ .
  - 3.1 Show that

$$E((U+1)^{-1}) = \ln(2)$$

and calculate

Var 
$$((U+1)^{-1})$$
.

Propose a Monte Carlo method for estimating In(2).

**3.2** Let 
$$f(t) = \frac{1}{t+1}$$
,  $t > -1$ . Compute

$$Cov(f(U), f(1 - U)).$$

3.3 Set

$$X = f(U)$$
 and  $Y = \frac{1}{2}(f(U) + f(1 - U))$ .

Show that Var(Y) < Var(X).

- **3.4** Based on the result in **3.3**, write an algorithm to estimate In(2). You may assume you have access to samples from the Uniform(0, 1) distribution. Compare the algorithm in **3.1** to the one derived in this item.
- **3.5** Generalize the results in the previous items to estimate ln(z), z > 1.