EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3021-WE01

Title:
Differential Geometry III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Let $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$ be the space curve

$$\boldsymbol{\alpha}(t) = (t, \sinh t, \cosh t).$$

1.1 Is $\boldsymbol{\alpha}$ smooth? Is $\boldsymbol{\alpha}$ regular? Is $\boldsymbol{\alpha}$ simple?

1.2 Compute the curvature and torsion of $\boldsymbol{\alpha}$.

1.3 Let $S = \{(x, y, z) \in \mathbb{R}^3 : z^2 - y^2 = 1, z > 0\}$. Show that S is a regular surface and that $\boldsymbol{\alpha}$ is a curve in S. Compute the geodesic curvature of $\boldsymbol{\alpha}$.

1.4 Using the parametrisation $\boldsymbol{x}(u, v) = (u, v, \sqrt{1 + v^2})$, compute the mean curvature of S.

Q2 **2.1** Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a smooth unit speed curve with

$$\boldsymbol{\alpha}'(s) = (\cos\theta(s), \sin\theta(s))$$

and assume that $\theta : I \to \mathbb{R}$ is also smooth. Find the curvature $\kappa : I \to \mathbb{R}$ of $\boldsymbol{\alpha}$ as an expression in the function θ.

2.2 Let $\boldsymbol{\beta} : I \to \mathbb{R}^2$ be a smooth unit speed curve with nowhere vanishing curvature function κ. Let $e : I \to \mathbb{R}^2$ be its evolute. Assume that e is constant, that is, there exists $\boldsymbol{p}_0 \in \mathbb{R}^2$ with $e(s) = \boldsymbol{p}_0$ for all $s \in I$. Show that the trace of $\boldsymbol{\beta}$ is contained in a circle centred at $\boldsymbol{p}_0$ and find the radius of this circle.

2.3 Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a smooth unit speed curve with principal normal $\boldsymbol{n}$ and binormal $\boldsymbol{b}$. Assume that there exist smooth functions $f, g : I \to \mathbb{R}$ with

$$\boldsymbol{\gamma}(s) = f(s)\boldsymbol{n}(s) + g(s)\boldsymbol{b}(s).$$

Show that there exists $c > 0$ such that $f^2(s) + g^2(s) = c$ for all $s \in I$. Express κ and τ in terms of f, g and g'. Show that $f(s) < 0$ for all $s \in I$ and that $\kappa(s) \geq 1/\sqrt{c}$ for all $s \in I$.

Q3 Consider the parametrised surface $\boldsymbol{x} : U = (0, 1) \times (0, 1) \to \mathbb{R}^3$,

$$\boldsymbol{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right).$$

3.1 Calculate the coefficients of the first fundamental form with respect to $\boldsymbol{x}$.

3.2 Calculate a Gauss map $\boldsymbol{N} : \boldsymbol{x}(U) \to S^2$.

3.3 Calculate the coefficients of the second fundamental form with respect to $\boldsymbol{x}$.

3.4 Calculate the principal curvatures and determine the umbilical points.

Q4 Consider the upper half plane model $\mathbb{H}^2 = \{(u, v) \in \mathbb{R}^2 : v > 0\}$ of the hyperbolic plane, with first fundamental form given by

$$E(u, v) = \frac{1}{v^2}, \quad F(u, v) = 0, \quad G(u, v) = \frac{1}{v^2}.$$

4.1 Describe the shape of all geodesics in $\mathbb{H}^2$ (you do not need to prove this).

4.2 Let $a < b$ and $c > 0$. Calculate the hyperbolic area of the subset

$$R := \{(u, v) \in \mathbb{H}^2 : a \leq u \leq b, v \geq c\}.$$

4.3 Calculate the geodesic curvature of the curve $\boldsymbol{\gamma} : [a, b] \to \mathbb{H}^2$, $\boldsymbol{\gamma}(t) = (t, c)$ for some constant $c > 0$.

4.4 State the Gauss-Bonnet Theorem and explain all involved terms. Using the fact that $\mathbb{H}^2$ has constant Gauss curvature -1 (without proof), verify in full detail that the Gauss-Bonnet Theorem holds for the set R.

Q5 Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a global parametrisation of a regular surface S and $\boldsymbol{N} : U \to S^2$ be a unit length normal, that is $\boldsymbol{N}(u, v) \perp T_{\boldsymbol{x}(u,v)}S$ and $\|\boldsymbol{N}(u, v)\| = 1$ for all $(u, v) \in U$. For every $t \in \mathbb{R}$, we consider the set $S_t \subset \mathbb{R}^3$, defined as the image of the global parametrisation

$$\boldsymbol{y}(u, v) = \boldsymbol{x}(u, v) + t\boldsymbol{N}(u, v).$$

We will assume that, for $|t| > 0$ small, the sets S_t are again regular surfaces.

5.1 Prove the identity

$$\frac{\partial \boldsymbol{N}}{\partial u} \times \frac{\partial \boldsymbol{N}}{\partial v} = K\, \boldsymbol{x}_u \times \boldsymbol{x}_v,$$

where K is the Gaussian curvature of S, viewed as a function on U.

5.2 Let E, F, G be the coefficients of the first fundamental form of $\boldsymbol{y}$. Express

$$\left.\frac{\partial E}{\partial t}\right|_{t=0}, \quad \left.\frac{\partial F}{\partial t}\right|_{t=0}, \quad \left.\frac{\partial G}{\partial t}\right|_{t=0}$$

in terms of the coefficients of the second fundamental form of $\boldsymbol{x}$.

5.3 Show that

$$\boldsymbol{y}_u \times \boldsymbol{y}_v = (1 - 2Ht + Kt^2)\boldsymbol{x}_u \times \boldsymbol{x}_v,$$

where H is the mean curvature of the surface S, viewed as a function on U.

5.4 Assume that the Gauss curvature K and the mean curvature H of S are globally bounded. Express, for small $|t| > 0$, the Gauss curvature of the regular surface S_t in terms of t, the Gauss curvature K and the mean curvature H of S.