Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3041-WE01

Title:
Galois Theory III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Find the degree $[E : \mathbb{Q}]$, where $E = \mathbb{Q}(\sqrt[8]{2}, \sqrt[11]{5})$.

1.2 Find a splitting field for the polynomial $F(X) = X^5 + X^4 + 1 \in \mathbb{F}_2[X]$.

1.3 Find all subfields in $\mathbb{F}_{3^{24}}$.

1.4 Find the minimal polynomial for $\alpha = \sqrt[3]{\sqrt{5} + 1}$ over $\mathbb{Q}$.

Q2 **2.1** Suppose $L = \mathbb{Q}(\zeta)$, where $\zeta \in \mathbb{C}$ is a primitive 80-th root of unity. Use ζ to find a primitive 5-th root of unity $\eta_1 \in L$ and a primitive 16-th root of unity $\eta_2 \in L$ such that $\zeta = \eta_1\eta_2$.

2.2 For $\eta_1, \eta_2 \in \mathbb{Q}(\zeta)$ from Question 2.1 find the minimal polynomials for ζ over $\mathbb{Q}(\eta_1)$ and over $\mathbb{Q}(\eta_2)$.

2.3 Find the Galois group of the polynomial $P(X) = X^3 - 2X + 2$ over $\mathbb{Q}(\sqrt{-19})$.

2.4 Let L be the cyclotomic field $\mathbb{Q}(\zeta_{532})$. How many different subfields $E \subset L$ such that $[E : \mathbb{Q}] = 6$ are there?

Q3 **3.1** Suppose F is a field of characteristic $\neq 2$ and E is a Galois extension of F such that $\mathrm{Gal}(E/F) = Z_2 \times Z_2$. Prove that there are $A, B \in F$ such that $E = F(\sqrt{A}, \sqrt{B})$.

3.2 Let L be a minimal normal over $\mathbb{Q}$ field extension of $\mathbb{Q}(\Theta)$, where $\Theta = \sqrt{8 - 5\sqrt{2}}$. Find $\mathrm{Gal}(L/\mathbb{Q})$.

3.3 For the field L from Question 3.2, find all subfields $K \subset L$ such that $[K : \mathbb{Q}] = 2$ and $\mathrm{Gal}(L/K) \simeq Z_2 \times Z_2$. Prove that there is a unique subfield K' of L such that $\mathrm{Gal}(L/K') \simeq Z_4$.

3.4 For each subfield K of the field L from Question 3.2 such that $\mathrm{Gal}(L/K) = Z_2 \times Z_2$, find $A, B \in K$ such that $L = K(\sqrt{A}, \sqrt{B})$.

Q4 **4.1** Suppose $L = \mathbb{F}_{11}(X)$ and $K = \mathbb{F}_{11}(X^5 + X^{-5})$, where X is a variable. Prove that L/K is Galois and describe the group structure of $\mathrm{Gal}(L/K)$.

4.2 Find all field extensions E of K in the field L from Question 4.1.

4.3 Explain a construction of the field extension $\mathbb{F}_8$ of $\mathbb{F}_2$ (define the corresponding set of elements and operations: addition and multiplication).

4.4 Find an irreducible polynomial $P(X)$ of degree 3 in $\mathbb{F}_8[X]$.

Q5 Let L be a minimal normal over $\mathbb{Q}$ field extension of $\mathbb{Q}(\sqrt[8]{2})$.

5.1 Find L.

5.2 Find generators and relations for $G = \mathrm{Gal}(L/\mathbb{Q})$.

5.3 Find all subfields $E \subset L$ such that $[E : \mathbb{Q}] = 2$.

5.4 Find the number of subfields E in L such that $\mathrm{Gal}(L/E)$ is cyclic of order 4.