Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3071-WE01

Title: Decision Theory III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Individuals Charles and Dora have utilities for positive amounts of money of the form $U_C(\$y) = y^\gamma$, $U_D(\$y) = y^\delta$, respectively, where $0 < \gamma < \delta$.

Discuss and compare the attitudes to risk of Charles and Dora.

[Any results that you require for each aspect of the discussion and comparison of risk attitudes should be stated clearly but need not be proved.]

1.2 Items of a particular type are rated by two attributes, quality, q, and reliability, r, which you consider to be mutually utility independent. You judge that the quality corresponds to your marginal utility for this attribute, i.e. that $U(q) = q$. You do not consider that r is expressed on your utility scale for reliability. Instead, you express the following isopreference curve between the two attributes:

$$r + q^3 = 1$$

Suppose that you are also indifferent between an item with rating $r = 0.5, q = 0.5$ and an item with rating $r = 0.6, q = 0.4$.

Evaluate your utility for items as a function of q and r.

1.3 Consider the following pay-off table for a two-person zero-sum game, where R chooses $R1$ or $R2$, and C chooses $C1, C2, C3$ or $C4$. The payoffs to R are as follows

	$C1$	$C2$	$C3$	$C4$
R1	0	3	5	1
R2	x	1	0	6

The payoff to C is minus the pay-off to R. Find all optimal strategies for R and the value of this game as function of x, for all values of $x > 0$ (not only the integers!). Find the optimal strategy for C for the case $x = 4$.

1.4 Consider the same game with the pay-offs specified in the table above, and let $x = 4$. Suppose now that both players aim at maximisation of their own expected pay-off. As they have no idea about the other player's strategy, they assign equal probabilities to each of the other player's options. Derive the optimal strategies for both players in this case, and use these to comment on any advantages of using the minimax criterion for such games.

Q2 A certain individual is known to have one of two variants V_1 or V_2 of a particular disease, with prior probability 0.5 for each variant. If the individual has variant V_1, then the appropriate medication treatment is M_1 and if the individual has variant V_2 then the appropriate medication is M_2. The individual cannot receive both medication treatments.

Suppose that the utility of receiving medication M_i if the individual has variant V_i is 10, and the utility of receiving M_i if the individual does not have variant V_i is zero.

Before selecting a medication, there is an option of administering a diagnostic test D_1 to the individual, at cost of U_1 utility units. The test gives two possible outcomes, namely it will give a positive indication for V_1 or a positive indication for V_2. If the patient has variant V_1, then the probability of obtaining a positive indication for V_1 is 3/4, and if the individual has variant V_2, then the probability of obtaining a positive indication for V_2 is 3/4.

After the outome of diagnostic test D_1 is revealed, there is an option of administering a second diagnostic test D_2 to the individual. This test costs U_2 units and cannot be administered unless test D_1 has already been made. This test also gives a positive indication for V_1 or for V_2. If the patient has variant V_1, then the probability of a positive indication for V_1 is 4/5, and if the patient has variant V_2, then the probability of a positive indication for V_2 is 4/5. Response to each test is independent given each variant.

2.1 Draw the decision tree for this problem.

2.2 Solve the tree for all values of U_1 and U_2, i.e. identify for which values of U_1 and U_2 you would make each combination of diagnostic tests, for which responses you would assign which medications and the overall utility of the best decision for each combination of utility costs.

2.3 For each combination of utility values, evaluate the risk profile for the optimal decision procedure. Comment on the risk profiles. Discuss the relevance of the analyses in this question to the decision choice for the individual.

Q3 In a certain production process, each item that is produced is acceptable with probability w, and unacceptable with probability $1 - w$, where $0 < w < 1$.

It is required to produce an estimate d for w with loss function

$$L(w, d) = \frac{1}{w}[c(w - d)^2 + d^2]$$

where c is a positive constant.

The prior distribution for w is a beta distribution, with parameters $\alpha > 1, \beta > 1$.

[The pdf of the beta distribution is

$$p(w) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1}(1-w)^{\beta-1}]$$

3.1 Find the Bayes rule and Bayes risk for this prior distribution. Discuss the behaviour of the Bayes rule as c varies.

3.2 Suppose that a sample of n independent items is inspected. The number of acceptable items is X with the remaining $n - X$ judged unacceptable. Find the Bayes rule and risk if the observed value of X is k acceptable items. Discuss the behaviour of the Bayes rule as c and n vary.

3.3 Find the Bayes risk of the sampling procedure, evaluated prior to sampling.

3.4 Suppose that an alternative estimator for w, namely $\overline{d} = \frac{X}{n}$ is being considered. Find the risk of $\overline{d}$. Compare the risk of $\overline{d}$ with the Bayes risk of the Bayes rule, for large n.

Q4 Ulrich and Vera are considering a bargaining problem with four options, for which their individual utilities are as follows

	A	B	C	D
Ulrich	10	0	9	6
Vera	0	5	1	3

They decide that, if they fail to reach agreement, they will settle for a fifth option, for which they both have utility 1.

4.1 Identify the Pareto boundary and the status quo point for this problem.

4.2 Find the Nash point and the equitable distribution point for this bargaining problem. Specify in detail what Ulrich and Vera should do corresponding to each of these two solution points.

4.3 While Vera's true utility for the fifth option is 1, she wonders if she can manipulate the resulting bargain according to the equitable distribution point by stating a utility $y \geq 1$ for this option. Investigate if this is possible by deriving the equitable distribution point as a function of y, assuming that Ulrich's utility for this option remains 1. Discuss briefly whether or not Vera can benefit from reporting any value $y > 1$.

4.4 Suppose that, instead of following the suggested actions corresponding to the Nash point or the equitable distribution point, Ulrich and Vera choose to present their problem, and the utilities above, to Leo, and want him to decide. Leo decides to use the following procedure to solve this problem:

'Maximum Sum of Utilities (MSU) Procedure':
Individually for Ulrich and Vera, Leo applies a positive linear transformation on their utilities, such that the least preferred option per person gets utility 0, and the most preferred option per person gets utility 1. Then, Leo sums up Ulrich's and Vera's scaled utilities per option, and decides that the option with maximum sum of such scaled utilities is the solution to this bargaining problem. If this procedure does not give a unique optimal solution, Leo picks one of the optimal solutions randomly, each with equal probability.

Solve Ulrich and Vera's bargaining problem using the MSU procedure. Analyse this MSU procedure, in general, from the perspective of Nash bargaining theory and the theory underlying the equitable distribution point. So you should explain whether or not this MSU procedure satisfies each of the axioms for bargaining problems corresponding to the Nash point and the equitable distribution point.

Q5 Five people, denoted A–E, have individual preference orderings over five options, denoted *a*–*e*, as given in the table below, where 1 indicates the most preferred option.

	1	2	3	4	5
A	*a*	*b*	*c*	*d*	*e*
B	*a*	*d*	*e*	*b*	*c*
C	*b*	*d*	*c*	*e*	*a*
D	*e*	*d*	*a*	*c*	*b*
E	*d*	*a*	*c*	*e*	*b*

They agree to use the following general procedure for combining their individual preferences into a group preference ordering:

Each option gets a score $s(\cdot)$, which is equal to the number of other options it beats in pairwise comparisons, using the majority rule and the individual preference orderings. For example, $s(b) = 1$ as option *b* only beats option *c* in pairwise comparison. The group preference ordering is simply based on those scores with e.g. $s(x) > s(y)$ indicating that the group prefers option *x* over option *y*, and $s(x) = s(y)$ indicating indifference between these two options. (Note that, for the general procedure, individuals are allowed to include indifferences between options in their individual preference orderings.)

5.1 Apply this procedure to the preference orderings in the table above, to derive the group preference ordering.

5.2 Explain whether or not this procedure satisfies each of the axioms in Arrow's Impossibility Theorem. Using the problem with the preference table above, give an example for each axiom which is not satisfied.

5.3 Before the preferences were revealed, someone feared that some other people might form a coalition and agree to each provide the same preference ordering. It is agreed to change the procedure as follows: if two or more individual preference orderings are identical, only one of these is included in the above procedure to derive a group preference order. Discuss briefly why the possibility of such a coalition could be problematic, whether or not this solution prevents this effectively, and explain whether or not this change to the procedure affects which axioms in Arrow's Impossibility Theorem are satisfied.

5.4 In Harsanyi's theorem of Utilitarianism, it is shown that the planner must sum up the individual utilities, scaled to $[0, 1]$, in order to derive a combined utility function for the group which satisfies two conditions, Anonymity and the Strong Pareto principle. In the same setting, one could also consider scaling the individual utilities to $[1, 2]$ and combining them by multiplication. Show that the result of this multiplication satisfies these two conditions, and explain whether or not this contradicts Harsanyi's theorem.
Hint: for this explanation, you may wish to consider combining the utilities of two people for the gamble $G = \frac{1}{2}V \oplus \frac{1}{2}W$, where both people consider V to be the least preferred option and W the most preferred option.