Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3091-WE01

Title:
Dynamical Systems III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Consider the 2×2 matrix

$$A = \begin{pmatrix} 0 & 9 \\ -1 & 6 \end{pmatrix} .$$

Determine the exponential $\exp(\lambda A)$, where λ is a real number.

1.2 Use the above result to obtain the solution, $\boldsymbol{x}(t)$, of the linear two-dimensional dynamical system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ with initial condition $\boldsymbol{x}(0) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.

1.3 Draw the corresponding phase flow. Explain your answer.

1.4 For the following choices of 2×2 matrices B, decide if the linear dynamical system $\dot{\boldsymbol{x}} = B\boldsymbol{x}$ is topologically conjugate to $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ and explain your reasoning

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} , \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

1.5 For a linear dynamical system $\dot{\boldsymbol{x}} = D\boldsymbol{x}$, consider a map $\boldsymbol{y} = \xi(\boldsymbol{x}) = e^C\boldsymbol{x}$ with a matrix C that satisfies $CD = DC$. Show that this implies $\dot{\boldsymbol{y}} = D\boldsymbol{y}$.

Q2 Consider the two-dimensional dynamical system

$$\dot{x} = 3x + 2y + 2y^3 , \quad \dot{y} = -2x - 3y - 2x^3 .$$

2.1 Explain why the origin is a hyperbolic critical point.

2.2 Employ the Picard iteration scheme to find an equation for the stable and unstable manifolds respectively, in the vicinity of the origin at the leading non-trivial order.

2.3 State the stable manifold theorem and verify that your solution in **2.2** is consistent with it.

2.4 Draw a potential phase diagram in your favourite coordinate system using *only* the results of questions **2.1**-**2.3**. Explain your answer.

Q3 The origin, $(x, y) = (0, 0)$, is the unique critical point of the two-dimensional dynamical system

$$\dot{x} = -y - x^3 , \quad \dot{y} = x^5 .$$

3.1 Find a Lyapunov function in a domain that contains the origin.

3.2 Prove that the origin is asymptotically stable. You may quote known theorems, but you need to explain how the assumptions of these theorems are satisfied in your argument.

Q4 **4.1** In each of the following three cases, give an explicit example of a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ which has an orbit $\varphi(t, \boldsymbol{x})$ such that:

(a) For all $p \in \varphi(t, \boldsymbol{x})$, $\omega(p)$ is a periodic orbit.

(b) For all $p \in \varphi(t, \boldsymbol{x})$, $\omega(p)$ consists of two heteroclinic orbits and two fixed points.

(c) For all $p \in \varphi(t, \boldsymbol{x})$, $\alpha(p) = \omega(p) = \boldsymbol{x}_0$, but $\varphi(t, \boldsymbol{x}) \neq \boldsymbol{x}_0$.

4.2 Now consider a two-dimensional dynamical system of the form

$$\dot{\boldsymbol{x}} = \nabla f(\boldsymbol{x})$$

and assume that D is a compact and positively invariant region that contains a single fixed point $\boldsymbol{x}_0$.

(d) Show that for every point $\boldsymbol{x} \in D$, $\omega(\boldsymbol{x}) = \boldsymbol{x}_0$.

(e) Show that $\lim_{t\to\infty} \varphi(t, \boldsymbol{x}) = \boldsymbol{x}_0$ for all $\boldsymbol{x} \in D$.

Q5 **5.1** For the dynamical system

$$\dot{x} = \sin(\pi x)$$
$$\dot{y} = \sin(\pi y)$$

find the Poincaré index for a curve γ that follows a square with corners at $(\pm 3/2, \pm 3/2)$ in counter-clockwise direction.

5.2 Sketch the flow in phase space of the two-dimensional dynamical system

$$\dot{x} = x(x^2 - \mu)$$
$$\dot{y} = -y$$

for $\mu > 0$. Using the properties of the Poincaré index, explain why the unique hyperbolic fixed point present for $\mu < 0$ is necessarily a saddle.

5.3 Use the Poincaré index for an appropriate dynamical system to show that a complex polynomial of degree n

$$P(z) = \sum_{i=0}^{n} c_i z^i$$

without multiple roots has n roots in the complex plane.

5.4 Prove that a dynamical system defined on the two-sphere S^2 cannot have a single hyperbolic fixed point.