Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3111-WE01

Title:
Quantum Mechanics III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

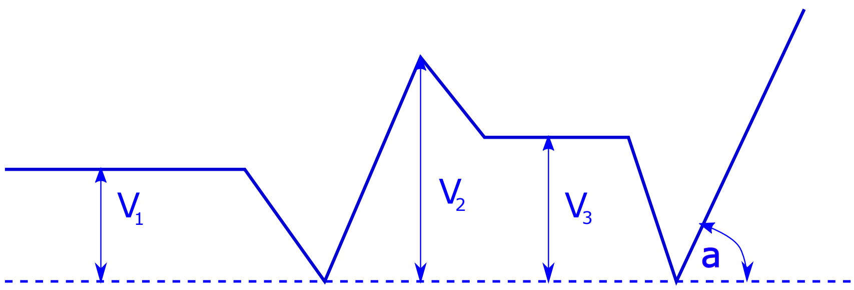
Q1 Consider a particle of mass m moving in one dimension inside the potential $V(x) = -\alpha x$.

1.1 Write an equation that expresses the time derivative of the expectation values $\langle \hat{p} \rangle_t$ and $\langle \hat{p}^2 \rangle_t$ in terms of $\langle \hat{p}^n \rangle_t$ where n is in each case a suitably chosen non-negative integer.

1.2 Write an equation that expresses the time derivative of $\langle \hat{p}^m \rangle_t$, for any positive integer m, in terms of $\langle \hat{p}^n \rangle_t$ where n is suitably chosen.

1.3 At $t = 0$ the particle is in a state $|\psi\rangle$ such that $\langle \hat{p} \rangle_{t=0} = p_1$ and $\langle \hat{p}^2 \rangle_{t=0} = p_2$. Solve the equations you found in question **1.1** to express $\langle \hat{p} \rangle_t$ and $\langle \hat{p}^2 \rangle_t$ in terms of p_1 and p_2.

A quantum particle is in a one dimensional potential shown on the image, where the values of the potential V_1, V_2 and V_3 satisfy the relation $V_1 < V_3 < V_2$ and the potential $V = V_1$ for $x \to -\infty$ and $V \to \infty$ as $x \to \infty$.



Answer the following questions:

1.4 Is the energy spectrum of the particle in this system discrete or continuous? If your answer depends on the energy of the particle, please explain and state which part of the spectrum is discrete and which continuous.

1.5 What is the qualitative form of the wave function of the particle? Sketch the wave function for the semi-classical particle in this potential and explain its qualitative behaviour, including the qualitative behaviour of the amplitudes and frequencies in various regions.

1.6 What would happen with the energy spectrum if the angle $a \to 0$?

1.7 What would happen with the energy spectrum if $V_1 \to \infty$?

Q2 Consider a Quantum Mechanical system with a Hilbert space spanned by the orthonormal basis $B = \{|0\rangle, |1\rangle\}$. The Hamiltonian of the system is

$$\hat{H} = E_0 \left(|0\rangle\langle 0| + 4\,|1\rangle\langle 1| + 2i\,|0\rangle\langle 1| + x\,|1\rangle\langle 0| \right),$$

with $x \in \mathbb{C}$ and $E_0 \in \mathbb{R}$. Moreover, our system has an observable which is described by the operator

$$\hat{O} = |0\rangle\langle 1| + |1\rangle\langle 0|\,.$$

2.1 Fix the complex number x, find the eigenvalues of $\hat{H}$ and write its eigenvectors in the basis B.

2.2 Use the results of the previous question to write the operator $e^{-it\hat{H}/\hbar}$ in terms of the basis B and its dual.

2.3 At $t = 0$ the system is in the initial state $|\psi(t = 0)\rangle = |0\rangle$. Write down the time evolved state $|\psi(t)\rangle$ after time t and the corresponding, time dependent, expectation value $\langle \hat{O} \rangle_t$.

Q3 Imagine a particle of mass m in a one dimensional simple harmonic oscillator potential of frequency ω. We are interested in the observable defined by,

$$\hat{\Omega} = Z \left(\hat{a}^\dagger\right)^2 \hat{a}^2 ,$$

with Z a real number. The operator $\hat{a}$ is defined in terms of the position $\hat{x}$ and momentum $\hat{p}$ operators according to,

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}} \left(m\omega\hat{x} + i\,\hat{p}\right) ,$$

and the Hamiltonian is

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) .$$

A complete set of Hamiltonian eigenstates is given by

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle$$

with $\hat{a}|0 >= 0$.

3.1 Write $\hat{\Omega}$ in terms of the operator $\hat{N} = \hat{a}^\dagger \hat{a}$ and show that they have common eigenvectors with $\hat{H}$. Determine the eigenvalue Ω_n of $\hat{\Omega}$ corresponding to its eigenstate $|n\rangle$.

3.2 Write the most general state $|\psi\rangle$ that can yield only zero in a measurement of $\hat{\Omega}$ in terms of the basis vectors $|n\rangle$. Moreover, you are provided with the information that all the possible values that this state can yield in an energy measurement are equally likely.

3.3 For the $|\psi\rangle$ of question **3.2** you are also given that it has expectations value $\langle \hat{x} \rangle = x_0$. Determine the possible expectation values of $\langle \hat{p} \rangle$ in terms of x_0.

3.4 Consider the state of **3.2** to be the initial state in time evolution under $\hat{H}$. Determine the expectation value $\langle \hat{\Omega} \rangle_t$ after time t. Justify your answer.

Q4 A quantum particle with mass m is oscillating in a k-dimensional space under the effect of the potential

$$V(x_1, \ldots, x_k) = \alpha(x_1^2 + x_2^2 + \ldots + x_{k-1}^2 + x_k^2) , \qquad \alpha > 0 .$$

This system is perturbed by adding the following extra terms to the potential,

$$V'(x) = \beta_1(x_1 + x_2) + \beta_2(x_1 + x_2)^2 .$$

Answer the following questions:

4.1 What is the eigen-energy spectrum of states and what are the eigen-energies of this system for the generic value of parameter k, in the absence of the perturbations? What is the number of eigen-energy states at the n-th excited level for this system?

4.2 When $k = 2, \beta_1 \neq 0, \beta_2 \neq 0$ what is the correction the ground state and its energy at first order in perturbation theory?

4.3 When $k = 2, \beta_1 \neq 0, \beta_2 \neq 0$, what is the correction to the energy of the second excited state in perturbation theory ?

4.4 When $k = 2, \beta_1 \neq 0, \beta_2 \neq 0$, write down the matrix from which you would compute the corrections to the energy of the n-th excited state in perturbation theory and find the correction to the energy if $\beta_2 = 0$.

Q5 **5.1** A quantum particle of mass m is bouncing inside the potential $V(x)$

$$V(x) = \begin{cases} ax^2 & x < 0 \\ 0 & 0 \leq x \leq b \\ x - b & b < x \end{cases}$$

where a, b are two positive constants. Write down the equation which fixes the allowed energies of the particle in the WKB approximation. You do not need to solve this equation.

5.2 Evaluate the following expressions

$$[\epsilon_{ijq}\hat{L}_j\hat{L}_q, L_p]$$
$$[\hat{x}_i\hat{x}_j, \hat{L}^2]$$
$$[\hat{x}_i\hat{p}_i, \hat{L}_z]$$
$$[\hat{L}_j, \hat{x}_i\hat{L}_i]$$

and write them in the simplest form. Start from the basic commutation relations between the angular momentum operators $\hat{L}_i$ ($i = 1, 2, 3$), as well as the basic commutation relations between the operators $\hat{p}_i$ and $\hat{x}_i$.