Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3141-WE01

Title:
Operations Research III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Is the statement below between quotes true or false? If true, provide a rigorous proof. If false, provide a fully worked counterexample. In your answer, you may use any results from the lectures, as long as you clearly state them.

"Consider the following two linear programming problems:

$$\begin{array}{cc} \max c^T x & \min b^T y \\ \text{where } Ax = b & \text{where } A^T y \geq c \\ x \geq 0 & y \text{ free} \end{array}$$

Then for any x satisfying $Ax = b$ and $x \geq 0$, and any y satisfying $A^T y \geq c$, we have that x and y are optimal solutions to the left and right problems, respectively, if and only if $x^T(A^T y - c) = 0$."

1.2 A factory receives wires of length 1, which may be cut at three predetermined positions

$$0 < d_1 < d_2 < d_3 < 1$$

along their lengths. Cuts can be made at none, one, two or all of these positions, but nowhere else (d_1, d_2, d_3 are fixed and cannot be changed). A piece of wire length ℓ can be sold on by the factory for a price of $\ell^2 + \frac{1}{2}$. Formulate and solve a dynamic programming problem to work out in which position(s) the factory should cut the wires, in order to get the maximum possible income. Identify all optimal solutions if there is more than one.

Q2 A fellow student is trying to solve a network problem, but has only seen the simplex method and none of the other methods yet. She has written down the following initial table:

T_0	v	f_{12}	f_{13}	f_{23}	f_{24}	f_{34}	s_{12}	s_{13}	s_{23}	s_{24}	s_{34}	
z	-1	0	0	0	0	0	0	0	0	0	0	0
?	-1	1	1	0	0	0	0	0	0	0	0	0
?	0	-1	0	1	1	0	0	0	0	0	0	0
?	0	0	-1	-1	0	1	0	0	0	0	0	0
?	0	1	0	0	0	0	1	0	0	0	0	6
?	0	0	1	0	0	0	0	1	0	0	0	1
?	0	0	0	1	0	0	0	0	1	0	0	4
?	0	0	0	0	1	0	0	0	0	1	0	2
?	0	0	0	0	0	1	0	0	0	0	1	3

(1)

The question marks indicate that she has not yet been able to identify an initial basic feasible solution.

2.1 Identify and draw the network for the problem that she is trying to solve. Interpret the numbers 6, 1, 4, 2, and 3, in the right hand side of the table.

2.2 On closer inspection, it might appear that the student has forgotten to include the following row into the above table:

T_0	v	f_{12}	f_{13}	f_{23}	f_{24}	f_{34}	s_{12}	s_{13}	s_{23}	s_{24}	s_{34}	
?	1	0	0	0	-1	-1	0	0	0	0	0	0

(2)

However, she had an excellent reason for not including this row. Explain why indeed this row must not be included.

2.3 With as little effort as possible, identify a starting basis for this problem. Derive the simplex table for your choice of basis.

2.4 After a number of pivoting steps, the student found the following simplex table:

T_*	v	f_{12}	f_{13}	f_{23}	f_{24}	f_{34}	s_{12}	s_{13}	s_{23}	s_{24}	s_{34}	
z	0	0	0	0	0	0	0	0	0	1	1	5
v	1	0	0	0	0	0	0	0	0	1	1	5
f_{24}	0	0	0	0	1	0	0	0	0	1	0	2
f_{34}	0	0	0	0	0	1	0	0	0	0	1	3
s_{12}	0	0	0	0	0	0	1	1	0	-1	-1	2
f_{13}	0	0	1	0	0	0	0	1	0	0	0	1
s_{23}	0	0	0	0	0	0	0	1	1	0	-1	2
f_{12}	0	1	0	0	0	0	0	-1	0	1	1	4
f_{23}	0	0	0	1	0	0	0	-1	0	0	1	2

(3)

Depict the optimal solution from this table on the network.

2.5 Comment on the uniqueness of the solution. If the solution is not unique, identify all solutions.

2.6 Apply the standard network algorithm from the notes on this network, and verify that your solution corresponds to what was found with the simplex method.

Q3 Vaccines produced at factories $i \in \{1, 2\}$ must be transported to cities $j \in \{1, 2, 3, 4\}$, with transportation costs, supplies, and demands respectively given by:

$$[c_{ij}] = \begin{bmatrix} 8 & 4 & 2 & 2 \\ 8 & 1 & 5 & 4 \end{bmatrix} \qquad [a_i] = \begin{bmatrix} 45 \\ 30 \end{bmatrix} \qquad [b_j] = \begin{bmatrix} 35 & 5 & 15 & 20 \end{bmatrix}$$

3.1 Use the North-West corner method to find an initial basic feasible solution.

3.2 Find the optimal transportation scheme.

3.3 Calculate the cost of your optimal solution.

Demand doubles in every city, but production remains the same, and every unit of unmet demand now incurs a cost of 15.

3.4 Suggest a good initial basic feasible solution for the new problem. Additional marks are given for identifying an initial basic feasible solution that is likely to save you some pivots, along with an explanation.

3.5 Find the optimal transportation scheme for the new problem.

3.6 Is your solution reasonable from the perspective of an inhabitant of city $j = 1$? If so, argue why. If not, suggest a change in the formulation of the linear program that would likely lead to a more reasonable solution.

Q4 **4.1** Write down the policy improvement algorithm for maximising the long-run expected reward per unit time for a Markov decision process on a finite state space.

Show that this algorithm does improve policies (i.e., the long-run expected reward per unit time increases, or stays the same, with each step of the algorithm).

You may assume that for any policy, if $x_i(n)$ is the expected total cost over the first n stages starting from state i and λ is the long-run expected reward per unit time, we have that for each state i, $x_i(n) - n\lambda$ converges to a finite limit as $n \to \infty$.

4.2 A dairy enthusiast has a fridge that can hold up to two bottles of milk. At the beginning of each day they must decide whether to go to the shop and buy a bottle of milk at a cost of *\$c*, or wait in for their neighbour to come around, who will give them a bottle of milk for free with probability $1/2$, otherwise give them nothing at all. If they wait for the neighbour they cannot go to the shop that day, and if they go to the shop then they will miss the neighbour's visit. If the fridge is full in the morning, they will not go to the shop and will have to reject any milk brought by the neighbour (otherwise they always accept it).

After this, the working day begins and the enthusiast will require $0, 1$ or 2 bottles of milk during the day with respective probabilities $1/2, 1/4, 1/4$. The oldest milk will always be used first. At the end of the day, any milk that has been in the fridge for three working days (i.e. milk that was donated/purchased in the morning two days previously) has to be wasted at a cost of *\$w*. If there is not enough supply to meet demand, this incurs a cost of *\$d* per bottle of milk short.

Formulate this as a Markov decision process, identifying the states and possible actions, as well as transition matrices and expected costs.

4.3 Under the policy of going to the shop unless the fridge is full, determine the long-run expected cost per day (in terms of c, d, w).

4.4 Suppose that $c = 3, w = 6$ and $d = 12$. Starting from the policy above, perform 2 steps of the policy improvement algorithm. Has an optimal policy been found?

Q5 A company has N candidates waiting to be interviewed for a position. The candidates are ordered uniformly at random (each possible ordering has equal probability), and then interviewed one by one according to this order. At the end of each interview the company must decide whether to accept or reject the candidate. At this point, the company can rank the candidates they have seen so far, but do not have any information about how they compare to the yet-to-be-interviewed candidates. The company must accept exactly one candidate and decisions cannot be overturned (in particular, if an "accept" decision is made, the process stops). The value of the jth best candidate to the company is $N - j$.

5.1 Suppose that the company wishes to maximise the expected value of the chosen candidate. Using stochastic dynamic programming, determine all possible optimal strategies when $N = 4$, and the maximum expected value.

5.2 For general N and $1 \le j \le n \le N$ define $E(j, n)$ to be the expected value to the company of the candidate interviewed at stage n, given they are the jth best seen so far (i.e., among the first n). Show that

$$E(j, n-1) = \frac{j}{n} E(j+1, n) + \frac{n-j}{n} E(j, n)$$

for all j, n. Use this to derive an exact expression for $E(j, n)$ in terms of j, n and N.

5.3 Show that for $n < N$, the company should never accept a candidate at stage n if $(n+1)/2$ or more better candidates have been seen beforehand.