Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3171-WE01

Title:
Mathematical Biology

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 1.1 Three-dimensional radial diffusion The radially symmetric diffusion of a chemical concentration, $c(r,t)$, in three dimensions is described by the nonlinear model

$$\frac{\partial c}{\partial t} = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2 c\frac{\partial c}{\partial r}\right), \tag{1}$$

where D is a positive constant, r is the spherical radial coordinate, and t is time. We assume that at $t=0$, the concentration is given by a delta function at the origin, and that the concentration must eventually go to zero in all directions as $r \to \infty$.

(i) Show that N, defined for $t > 0$ as

$$N = 4\pi \int_0^\infty r^2 c \, \mathrm{d}r,$$

is constant and interpret this statement. You can assume that $c\dfrac{\partial c}{\partial r} \to 0$ faster than $r^2 \to \infty$.

(ii) Consider a similarity solution of the form

$$c(r,t) = \frac{1}{t^\alpha} v(\eta), \qquad \eta = \frac{r}{t^\beta},$$

where α and β are to be determined. By substituting this into (1) and subsequently finding α and β, show that $v(\eta)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[5D\eta^2 v\frac{\mathrm{d}v}{\mathrm{d}\eta} + \eta^3 v\right] = 0.$$

(iii) Show that there exists a value η_0 (which you do not need to find) such that $v(\eta) > 0$ for $\eta < \eta_0$, but $v(\eta) = 0$ for $\eta > \eta_0$. Interpret this finding, especially in comparison to the traditional diffusion equation $\partial c/\partial t = D\nabla^2 c$.

(iv) Determine the volume within which $c(r,t) > 0$ at a given time t in terms of η_0.

1.2 Non-Euclidean patterns Consider a domain which is a segment of an annulus. It is covered by polar coordinates (r,θ) with the coordinates taking values on $r \in [R_i, R_o]$ and $\theta \in [0, 3\pi/2]$. We consider the reaction–diffusion equations on this system for a pair of interactive (scalar) biological densities u, v:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla^2 u + \gamma F(u,v),\\ \frac{\partial v}{\partial t} &= D\nabla^2 v + \gamma G(u,v),\end{aligned} \tag{2}$$

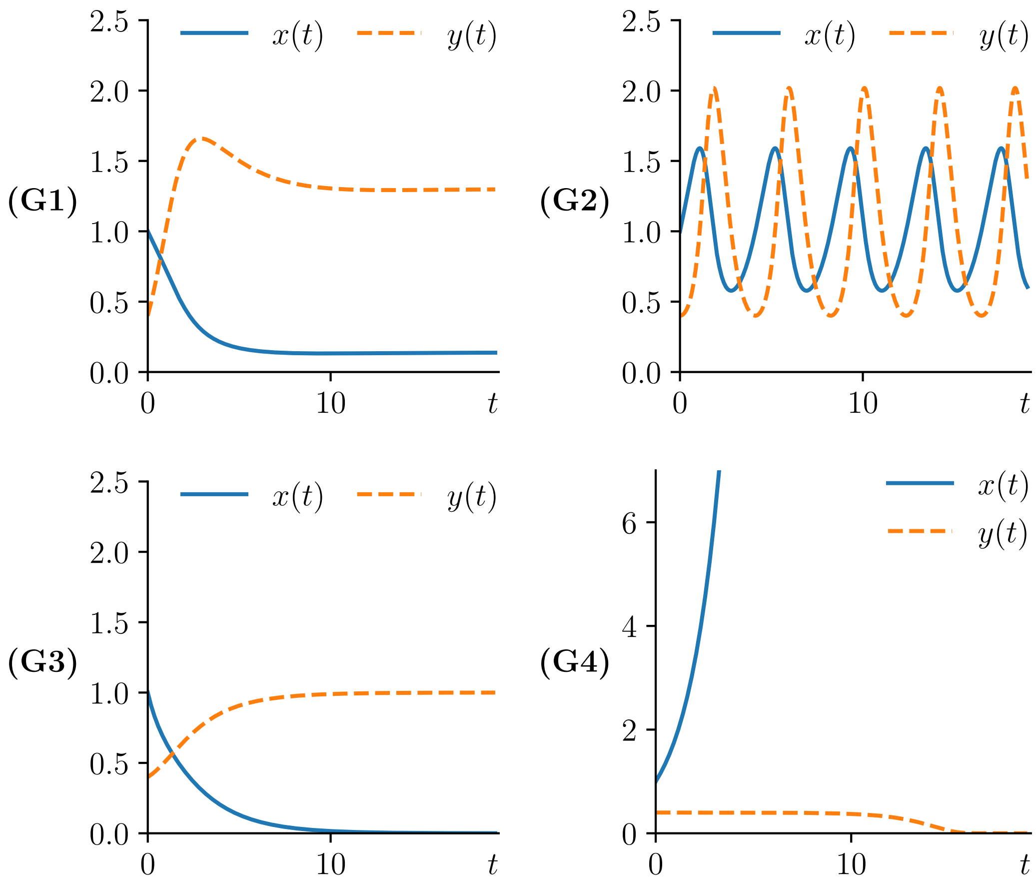
where $D, \gamma > 0$. We assume this system has a homogeneous equilibrium (u_0, v_0).

Assume $u(r,\theta,t) = u_0 + \epsilon u_1(r,\theta,t)$ and $v(r,\theta,t) = v_0 + \epsilon v_1(r,\theta,t)$. Find the general solutions to the $\mathcal{O}(\epsilon)$ expansion of system (2). You do not need to write the solution as a single whole sum, just enumerate all possibilities. You may use any results derived in class.

Q2 Variations on Lotka–Volterra

2.1 The following graphs plot the populations of two species, $x(t)$ and $y(t)$, over time, t. Each graph, labelled **(G1)–(G4)**, has the same initial condition and corresponds to one of the nondimensionalised models listed below, labelled **(M1)–(M4)**. Giving a good reason in each case, match the graphs to the models.

Graphs:



Models:

(M1)
$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x - xy \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = \gamma(-y + xy) \end{cases}$$

(M2)
$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x - xy \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = \gamma(-y - xy) \end{cases}$$

(M3)
$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x) - \gamma xy \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = \delta y(1 - y) + \beta xy \end{cases}$$

(M4)
$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x) - \gamma xy \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = \delta y(1 - y) - \beta xy \end{cases}$$

where the constants $\beta, \gamma, \delta > 0$ wherever they appear, although they are not necessarily equal between graphs.

2.2 In the traditional Lotka–Volterra system with two populations, one species is the food source of the other. In this part of the question, we will consider two populations, represented by $x(t)$ and $y(t)$, competing for the *same* food source. In nondimensional form, this system can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + \alpha x R(x,y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\beta y + \gamma y R(x,y),$$

where

$$R(x,y) = R_0 - cx - dy,$$

and α, β, γ, c, d and R_0 are positive constants. We will assume that $1/\alpha \neq \beta/\gamma$.

(i) Give a thorough biological interpretation for the terms in this system.

(ii) Find the three equilibria of the system and the conditions under which they exist.

(iii) By performing a linear stability analysis on the equilibria, show that if two competing populations depend on the same food source, at least one of them will become extinct. As part of your analysis, you should show that the asymptotic stability of one equilibrium is guaranteed, for any set of values of the constants in the system.

Q3 Species conservation A team of mathematical biologists are looking to model the population of fish in a section of a free-flowing river. They propose to model the section of river as a one-dimensional domain of length L, and to model the population number as a function of space and time, $u(x,t)$. Their chosen model is

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + au, \tag{3}$$

with boundary conditions

$$u(0,t) = u(L,t) = C, \tag{4}$$

where D, a and C are positive constants.

3.1 Comment on whether you think this model is appropriate. Is anything crucial missing? Provide at least four good arguments in your answer.

3.2 Choose an appropriate scaling for x and t in order to show that the nondimensionalised form of (3) and (4) is

$$\frac{\partial u}{\partial \widehat{t}} = \frac{\partial^2 u}{\partial \widehat{x}^2} + u, \qquad u(0,\widehat{t}) = u(\widehat{L},\widehat{t}) = C,$$

where hats represent nondimensional variables, and where you should specify the value of $\widehat{L}$ in terms of the original variables and constants.

The banks of the river are home to a small population of bears, who regularly feast on the fish. The team now decide to model the interacting population of fish, $u(x,t)$, and bears, $v(x,t)$, by the now nondimensionalised equations

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + u - \frac{uv}{1+u},\\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} - \beta v + \frac{uv}{1+u},\end{aligned}$$

(i.e. we have dropped hats). The boundary conditions are

$$u(0,t) = u(L,t) = C_1, \qquad v(0,t) = v(L,t) = 0,$$

where $C_1 \geq 0$.

3.3 Comment on whether you think this new model is appropriate. Provide at least four good arguments in your answer, at least one of which looks at the behaviour of the bears.

3.4 Show that there are two homogeneous equilibria to this system: one with both species extinct, and one with both species co-existing. Note the range of permitted β for the co-existence equilibrium to exist.

The team is given some data which suggests an upper bound of $\beta < 4/5$. They then perform a stability analysis on the homogeneous equilibria and find a restriction on L for stability of the co-existence equilibrium. They use this to argue that for the continued existence of both fish and bears along the river, the bears should be kept within a section of the river bank of a given length.

3.5 Do you agree with this conclusion? Perform your own stability analysis on the co-existence equilibrium. If you agree with the conclusion, state the form of a restriction on L. Otherwise, show that stability of this equilibrium is not dependent on L.

Q4 Turing analysis Consider the following nondimensionalised reaction–diffusion system for scalar densities u and v:

$$\frac{\partial u}{\partial t} = \nabla^2 u + \gamma F(u,v),$$
$$\frac{\partial v}{\partial t} = D\nabla^2 v + \gamma G(u,v),$$

where D and γ are positive constants. We assume a Cartesian domain $[0, L_1]\times[0, L_2]$ with coordinates (x_1, x_2). The boundary conditions are $u, v = 0$ on the boundaries of constant x_1 and no-flux on the boundaries of constant x_2. The Turing conditions for pattern formation are

$$\begin{aligned} &F_u + G_v < 0, \quad F_u G_v - G_u F_v > 0,\\ &G_v + DF_u > 0, \quad (G_v + DF_u)^2 - 4D(F_u G_v - G_u F_v) > 0, \end{aligned} \tag{5}$$

where we have used the notation

$$F_u = \left.\frac{\partial F}{\partial u}\right|_{u=u_0,v=v_0}, \quad F_v = \left.\frac{\partial F}{\partial v}\right|_{u=u_0,v=v_0},$$
$$G_u = \left.\frac{\partial G}{\partial u}\right|_{u=u_0,v=v_0}, \quad G_v = \left.\frac{\partial G}{\partial v}\right|_{u=u_0,v=v_0},$$

and (u_0, v_0) represents a homogeneous equilibrium of the system.

4.1 Give the explicit mathematical form for the pattern and describe what the conditions (5) enforce.

4.2 Describe why the following reaction functions,

$$F(u,v) = u^n + H_1(v), \quad G(u,v) = H_2(u) + v^m, \tag{6}$$

could not satisfy the Turing conditions for a pair of physical populations for which $u \geq 0, v \geq 0$. Here n, m are positive integers and H_1, H_2 are smooth real-valued functions. Give the physical interpretation of this failure.

4.3 Assume we kept the same forms of F and G as in (6), but allowed for a model in which u and v must take on negative values. What values of m, n would (potentially) allow for the Turing conditions

$$\begin{aligned} &F_u + G_v < 0, \quad F_u G_v - G_u F_v > 0,\\ &G_v + DF_u > 0, \end{aligned}$$

to be satisfied? What constraint on H_1, H_2 must apply for this to be possible?

4.4 Now consider the following variant of the reaction–diffusion system introduced at the beginning of the question,

$$\frac{\partial u}{\partial t} = \nabla^2 u - v^2 \int_0^t u \, \mathrm{d}t,$$
$$\frac{\partial v}{\partial t} = \nabla^2 v + G(v),$$

on a one-dimensional domain $[0, L]$ with coordinate x. Assume $u = u_0(t)$ is homogeneous and v is in homogeneous equilibrium (with value $v_0 > 0$). Show that if $u_0(0) = 0$ and $\mathrm{d}u_0/\mathrm{d}t(0) = v_0$, then

$$u_0(t) = \sin(v_0 t).$$

4.5 Assume pattern formation solutions in the form:

$$u(x,t) = \sin(v_0 t) + \epsilon \mathrm{e}^{\mathrm{i}kx} f(t), \quad v(x,t) = v_0 + \epsilon \mathrm{e}^{\mathrm{i}kx} g(t).$$

Show that to linear order the temporal functions satisfy the following system of ODEs:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= -k^2 f - v_0^2 \int_0^t f \, \mathrm{d}t + 2g(t)\cos(v_0 t), \\ \frac{\mathrm{d}g}{\mathrm{d}t} &= -(k^2 - G_v) g. \end{aligned} \tag{7}$$

4.6 Find a condition determining when a mode k will grow to form temporally oscillating patterns which are $\mathcal{O}(1)$, as governed by the system (7).

Q5 Planar elastic rods Consider a thin slender elastic body whose central axis is represented by a curve $\mathbf{r}(s) : [0, L] \to \mathbb{R}^3$, where s is the body's arclength. The coordinates of $\mathbb{R}^3$ take the form (x, y, z) with unit vectors $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{z}}$ respectively. A force $\mathbf{n}(s)$ represents the internal force acting on each cross-section of the body and $\mathbf{m}(s)$ represents the internal couple acting on each cross-section. Net forces and couples can be applied at $s = 0$ and $s = L$.

The general equilibrium for this body takes the following form:

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} + \mathbf{f} = \mathbf{0},$$
$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}s} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \times \mathbf{n} + \mathbf{l} = \mathbf{0},$$

where $\mathbf{f}$ is an external force per unit length acting on the body and $\mathbf{l}$ is an external couple per unit length acting on the body.

For all permissible configurations the curve $\mathbf{r}(s)$ can, for some vector $\mathbf{u}(s)$, be determined by solving the following system of differential equations

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{d}_3, \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{d}_j = \mathbf{u} \times \mathbf{d}_j, \quad \mathbf{u} = u_1\mathbf{d}_1 + u_2\mathbf{d}_2 + u_3\mathbf{d}_3,$$

(up to a choice of initial conditions). Here $\mathbf{d}_3$ is the unit tangent vector of $\mathbf{r}$, and $\mathbf{d}_1$ and $\mathbf{d}_2$ form an orthonormal frame with $\mathbf{d}_3$.

5.1 You are told that a collagen fibre (a thin tubular protein structure) stretches by up to a factor of 2 under typical skeletal pressures. Why would the slender body model described in the introduction not be appropriate for this collagen fibre?

5.2 Consider a particular case of the model for which $\mathbf{f} = \mathbf{l} = \mathbf{0}$. Assume further that $\mathbf{d}_3 = (\cos\theta, 0, \sin\theta)$, $\mathbf{d}_1 = (-\sin\theta, 0, \cos\theta)$ and that the moment takes the form:

$$\mathbf{m} = A(s)u_1\mathbf{d}_1 + A(s)u_2\mathbf{d}_2 + Cu_3\mathbf{d}_3,$$

where C is a constant and $A(s)$ a smooth function. Finally, assume a force $N\mathbf{d}_3$ is applied at $s = L$ where $\theta(L) = \theta_0$. Show that the equilibrium of this system reduces to the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[A(s)\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] - N\sin(\theta - \theta_0) = 0.$$

5.3 Assume the following potential forms of the bending function $A(s)$:

(i) $A(s) = A_0 + B_0\sin(2\pi ns/L)$ for n integer and $A_0, B_0 > 0$ constants with $B_0 < A_0$.

(ii) $A(s) = A_0 - B_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2$ with $A_0, B_0 > 0$ constants.

Describe the physical interpretation of each model choice and state any potential significant (physically unrealistic) flaws with the models, if there are any.

5.4 Assume $A(s) = A_0 - f(s)$ $(f(s) < A_0 \in [0, L])$. It can be shown that if

$$\mathbf{l} = -\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s}\widehat{\mathbf{y}},$$

the equilibrium reduces to the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[A(s)\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] - N\sin(\theta - \theta_0) + \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0. \tag{8}$$

Solve for $\theta(s)$ given the boundary conditions $\mathrm{d}\theta/\mathrm{d}s = 0$ when $\theta = \theta_0$ and $\theta(0) = 0$.

5.5 The solution to (8) has the property that it has infinite length (L is unbounded). We now allow $f(s)$ to be any smooth function. Suggest a form which $f(s)$ could take for the solution to have finite length, such that $\theta_0 > 0$, and give the physical interpretation of this assumption (with regards to the form of $A(s)$).