Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2021 | MATH3201-WE01 |

| Title: |
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| Geometry III |

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| Time (for guidance only): | 3 hours | |
| Additional Material provided: | Formula sheet | |
| Materials Permitted: | | |
| Calculators Permitted: | Yes | Models Permitted: There is no restriction on the model of calculator which may be used. |

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| Instructions to Candidates: | Credit will be given for your answers to all questions.<br>All questions carry the same marks.<br><br>Please start each question on a new page.<br>Please write your CIS username at the top of each page.<br><br>To receive credit, your answers must show your working and explain your reasoning. |

| Revision: | |
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ED01/2021

**Q1** **1.1** Let $\triangle A_1A_2A_3$ be a triangle in $\mathbb{E}^2$, let $A_4 = A_1$. Let $B_i \in A_iA_{i+1}$, $i = 1, 2, 3$, be points on the sides of $\triangle A_1A_2A_3$ such that $\frac{B_iA_i}{B_iA_{i+1}} = 3$. Show that the medians of $\triangle B_1B_2B_3$ intersect at the same point as the medians of $\triangle A_1A_2A_3$.

**1.2** Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be circles in $\mathbb{E}^2$. Suppose that $\mathcal{C}_1$ does not intersect $\mathcal{C}_2$. Suppose $\mathcal{C}_3$ is orthogonal to both $\mathcal{C}_1$ and $\mathcal{C}_2$. Show that there exists a circle $\mathcal{C}_0$ orthogonal to all three circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$.

**1.3** Let $\mathcal{C}$ be a circle centred at $O \in \mathbb{E}^2$ and let $A, B \in \mathcal{C}$ be points such that $\angle AOB = \gamma < \pi$. Let $\mathcal{C}'$ be the circumscribed circle for $\triangle ABO$. Find the acute angle between $\mathcal{C}$ and $\mathcal{C}'$.

**1.4** Let $ABC$ be a spherical triangle such that $\angle A > \angle B$. Is it always true that $BC > AC$? Justify your answer.

**Q2** **2.1** Let $ABCD$ be a quadrilateral on the unit sphere. Let $\angle ABC = \angle CDA = \pi/2$ and $AB = BC = CD = DA = a$. Find $BD$.

**2.2** Show that there exists a spherical regular pentagon with all angles $2\pi/3$. Find its area.

**2.3** Let $ABC$ be a triangle on $S^2$ with three acute angles (i.e. $\angle A, \angle B, \angle C < \pi/2$). Let $\triangle A'B'C'$ be the polar triangle to $\triangle ABC$. Is it possible that $\triangle A'B'C'$ also has three acute angles? Justify your answer.

**2.4** Let $f$ be an isometry of $S^2$. Suppose that $f \circ r = r \circ f$ for every reflection $r$. Describe all such isometries $f$. Justify your answer.

**Q3** **3.1** In $\mathbb{R}P^2$, compute the cross-ratio $[A, B, C, D]$ of the points $A, B, C, D$ given by $(0 : 1 : 0)$, $(2 : 1 : 0)$, $(1 : 2 : 0)$, $(1 : 1 : 0)$ respectively.

**3.2** Four lines in $\mathbb{R}P^2$ are in general position if no three of them are concurrent. Is it true that the group of projective transformations acts transitively on all quadruples of lines in general position?

**3.3** Let $a_1, a_2, a_3, a_4$ be lines in $\mathbb{R}P^2$ concurrent at the point $A$, and let $b_1, b_2, b_3$ be lines concurrent at the point $B$. Denote $P_{ij} = a_i \cap b_j$ for $i = 1, \dots, 4$ , $j = 1, 2, 3$. Suppose that the lines $P_{12}P_{21}, P_{22}P_{31}, P_{32}P_{41}$ are concurrent. Show that the lines $P_{12}P_{23}, P_{22}P_{33}, P_{32}P_{43}$ are concurrent too.

**3.4** Formulate the statement dual to the statement in part (3.3) and prove it without referring to the proof of (3.3).

**Q4** Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be mutually tangent circles not passing through the same point and let $\mathcal{C}$ be a circle tangent to each of them.

**4.1** How many ways are there to chose a circle or line $\mathcal{C}$ so that $\mathcal{C}$ is tangent to each of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$?

**4.2** Assume the circle $\mathcal{C}_4$ is disjoint from $\mathcal{C}_2$ and tangent to $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}$. Let $P_i = \mathcal{C}_i \cap \mathcal{C}_{i+1}$ for $i = 1, 2, 3, 4$ (where $\mathcal{C}_5 := \mathcal{C}_1$).
Show that $[P_1, P_2, P_3, P_4]$ is real.

**4.3** Under the assumptions of (4.2), let $\mathcal{C}_{013}$ be a circle or line orthogonal to the circles $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_3$, and let $\mathcal{C}_{024}$ be a circle or line orthogonal to $\mathcal{C}, \mathcal{C}_2, \mathcal{C}_4$. Which values can the angle between $\mathcal{C}_{013}$ and $\mathcal{C}_{024}$ take?

**4.4** Under the assumptions of (4.2), is there a Möbius transformation which takes circles $\mathcal{C}_2$ and $\mathcal{C}_4$ to concentric circles of radius 1 and 2? Justify your answer.

**Q5** Let $ABCD$ be an ideal quadrilateral in $\mathbb{H}^2$.

**5.1** Does the group of isometries $Isom(\mathbb{H}^2)$ act transitively on ideal quadrilaterals?

**5.2** Let $l$ be a common perpendicular to $AB$ and $CD$. Show that the lines $AC, BD$ and $l$ are concurrent.

**5.3** Suppose that the quadrilateral $ABCD$ has an inscribed circle, i.e. a circle tangent to all four sides. Which values can the radius of the inscribed circle take?

**5.4** Assume that $ABCD$ has an inscribed circle. Let $r_1, r_2, r_3, r_4$ be reflections with respect to $AB, BC, CD, DA$. Find the type of the isometry $r_4 r_3 r_2 r_1$.