Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3211-WE01

Title:
Probability III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** For $a \in \mathbb{R}$ and $\gamma > 0$ a random variable X has CAUCHY(a, γ) distribution if it has the probability density function $f_X(u)$

$$f_X(u) = \frac{1}{\pi\gamma(1 + (\frac{u-a}{\gamma})^2)}, \qquad u \in \mathbb{R}.$$

If X is a CAUCHY(a, γ) random variable and Y is an independent CAUCHY(b, μ) random variable, what is the distribution of $X + Y$?

[*Hint*: you may use that the characteristic function of a CAUCHY(a, γ) random variable is $\varphi(t) = \exp(iat - \gamma|t|)$.]

1.2 A friend tells you they have found a wonderful new probability distribution MAGIC on $\mathbb{R}$ such that if X is a MAGIC random variable, then X has characteristic function $e^{-i|t|}$. Prove that your friend has made a mistake in their computations, i.e., that there cannot be any such random variable X.

1.3 Let $X \sim \text{Bin}(n_1, p_1)$ and $Y \sim \text{Bin}(n_2, p_2)$ for integers $1 \le n_1 \le n_2$ and reals $0 \le p_1 \le p_2 \le 1$. Are X and Y stochastically ordered? Justify your answer by proving the result or giving a counter-example.

1.4 Suppose further that $X \sim \text{Bin}(2, 1/4)$ and $Y \sim \text{Bin}(3, 1/2)$. Describe all monotone couplings $(\tilde{X}, \tilde{Y})$ of X and Y.

1.5 For the couplings described in part **1.4**, what is the maximum that $\mathbf{P}(\tilde{X} = \tilde{Y})$ can be?

For each problem you should show all your workings and justify your calculations with suitable explanations.

Q2 In each of the following problems, $(X_i)_{i=1}^{\infty}$ is a sequence of IID real-valued random variables.

[*Caution*: do not assume anything else about the X_i beyond what each individual problem states.]

2.1 Suppose that for all $x \ge 100$ the X_i satisfy $\mathbb{P}[X_i > x] = e^{-x}$. Find a constant c such that $\mathbf{P}[\limsup_{n\to\infty} \frac{X_n}{\log n} = c] = 1$.

2.2 True or false: there is a constant $c \in \mathbb{R}$ such that $\mathbf{P}[\limsup_{n\to\infty} \frac{X_n}{\log n} = c] = 1$.

2.3 Let $M_k = \max_{1\le i\le k} X_i$. State carefully what it means for $\frac{M_k}{\log k}$ to converge to a random variable Y almost surely.

2.4 Suppose that for all $x \ge 100$ the X_i satisfy $\mathbb{P}[X_i > x] = e^{-x}$. Let $M_k = \max_{1\le i\le k} X_i$. True or false: $\frac{M_k}{\log k}$ converges to a constant random variable almost surely.

[*Hint*: you may use the inequality $(1 - a)^b \le e^{-ab}$ for $a \in (0, 1)$ and $b > 0$.]

For each problem you should show all your workings and justify your calculations with suitable explanations.

Q3 Suppose $(X_i)_{i=1}^{\infty}$ is a sequence of IID random variables with X_i taking values in $\{1, 2, 3, \dots, K\}$ for some $K \in \mathbb{N}$. Let $N_k(j) = \sum_{i=1}^{k} 1_{\{X_i=j\}}$ denote the number of times that the first k variables X_i take the value j.

3.1 Suppose that $\mathbf{P}[X_i = 1] = \mathbf{P}[X_i = 2] = \cdots = \mathbf{P}[X_i = K] = 1/K$.

(i) Show that there exist constants $c_1, c_2 > 0$ independent of n such that for all n large enough

$$c_1 n^{(1-K)/2} \leq \mathbf{P}[N_{Kn}(1) = \cdots = N_{Kn}(K) = n] \leq c_2 n^{(1-K)/2}$$

(ii) Let $\tau = \inf\{n \geq 1 \mid N_{Kn}(1) = \cdots = N_{Kn}(K) = n\}$. Is $\mathbf{P}[\tau < \infty] = 1$?

3.2 Suppose that $\mathbf{P}[X_i = 1] = 1/2$ and $\mathbf{P}[X_i = 2] = \mathbf{P}[X_i = 3] = 1/4$.

(i) Let $\tau_1 = \inf\{n \geq 1 \mid N_{4n}(1) = 2N_{4n}(2) = 2N_{4n}(3)\}$. Is $\mathbb{P}[\tau_1 < \infty] = 1$?

(ii) Let $\tau_2 = \inf\{n \geq 1 \mid N_n(1) = N_n(2)\}$. Is $\mathbb{P}[\tau_2 < \infty] = 1$?
[*Hint*: you may use the conclusions of the exercises of the course in your solution without providing proofs. Clearly state any such conclusions that you use in this way.]

For each problem you should show all your workings and justify your calculations with suitable explanations.

Q4 **4.1** Let $X_1, X_2, \dots, X_n$ be independent $\mathcal{U}(0, 1)$ random variables. For $k = 1, \dots, n$, calculate the cumulative distribution function of the k-th order variable $X_{(k)}$, and prove that $X_{(k)}$ has a beta distribution.

4.2 For $1 < i < n$, calculate the conditional density $f_{X_{(i-1)},X_{(i+1)}|X_{(i)}}(x, z|y)$ of $X_{(i-1)}$ and $X_{(i+1)}$ given $X_{(i)}$.

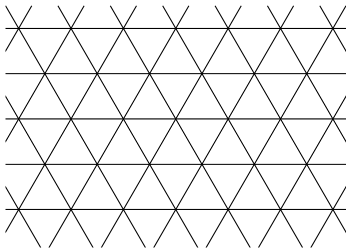
4.3 Using part **4.2** or otherwise, calculate the joint density $f_{\Delta_{(i)}X,\Delta_{(i+1)}X}(r, s)$ of the gaps $\Delta_{(i)}X = X_{(i)} - X_{(i-1)}$ and $\Delta_{(i+1)}X = X_{(i+1)} - X_{(i)}$. What can you say about the joint distribution of the rescaled gaps $n\Delta_{(i)}X$, $n\Delta_{(i+1)}X$ for large n?

4.4 Let $Y_1, \dots, Y_n$ be independent Exp(1) random variables, with order statistic $(Y_{(1)}, \dots, Y_{(n)})$. Prove that, for $1 \leq k < m \leq n$, the pair $(Y_{(k)}, Y_{(m)})$ has the same joint distribution as $(\sum_{j=1}^{k} \alpha_j Y_j, \sum_{j=1}^{m} \alpha_j Y_j)$, where the constants α_j are to be determined.

4.5 For fixed $0 < p < q < 1$, define $I_{p,q,n} := Y_{(\lfloor qn \rfloor)} - Y_{(\lfloor pn \rfloor)}$. Using part **4.4** or otherwise, prove that $I_{p,q,n}$ converges in probability to a constant $c \equiv c_{p,q}$, as $n \to \infty$, and give $c_{p,q}$ in terms of p and q.

For each problem you should show all your workings and justify your calculations with suitable explanations.

Q5 Consider bond percolation on the triangular lattice, pictured below, where each bond is open independently with probability $p \in [0, 1]$.



5.1 Carefully define the percolation probability $\theta_x(p)$, where x is a vertex of the lattice, and explain why it does not depend on the choice of x.

5.2 Prove that the function θ_x is non-decreasing. What does this imply about $p_{cr} := \inf\{p : \theta_x(p) > 0\}$?

5.3 Prove that $p_{cr} > 0$ and find an explicit value $p_1 > 0$ with $p_{cr} \geq p_1$.

5.4 Prove that $p_{cr} < 1$ and find an explicit value $p_2 < 1$ with $p_{cr} \leq p_2$.

For each problem you should show all your workings and justify your calculations with suitable explanations.