EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3231-WE01

Title:
Solitons III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 1.1 A field $u(x,t)$ has energy E and momentum P given by

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} dx \ \left[u_t^2 + u_x^2 + W'(u)^2\right] , \qquad P = -\int_{-\infty}^{+\infty} dx \ u_t u_x ,$$

where $W'(u) = dW(u)/du$ is the first derivative of $W(u)$, which is a function of u with isolated stationary points. Consider field configurations of the form $u(x,t) = f(x - vt)$ for a constant v such that $|v| < 1$. Find a lower bound for the linear combination $E - vP$ in terms of a conserved topological charge, which only depends on the asymptotic values of the field at spatial infinity.

1.2 Show that the lower bound for $E - vP$ is saturated if $u(x,t) = f(x - vt)$ obeys the equation of motion $u_{tt} - u_{xx} + W'(u)W''(u) = 0$ with appropriate boundary conditions, which you should derive from the data in question 1.1.

1.3 At $t = 0$, the associated linear problem for a localised solution $u(x,t)$ of the KdV equation has a single bound state, with unnormalised eigenfunction $\psi(x) = \exp(-2|x|)$.

(i) What is the corresponding normalisation constant c_1 in the scattering data for $u(x,0)$?

(ii) What does c_1 become a time t later, when $u(x,0)$ has evolved into $u(x,t)$? Justify your answer briefly with reference to the asymptotic behaviour of the B operator in the Lax pair for the KdV equation.

1.4 (i) Define the Wronskian $W[f,g]$ between two functions $f(x)$ and $g(x)$, and show that if f and g are linearly dependent then $W[f,g] = 0$.

(ii) Show that if **(a)** $W[f,g] = 0$ on some interval I, and **(b)** at least one of f and g is never zero on I, then f and g are linearly dependent on I.

(iii) Find a pair of infinitely-differentiable functions for which condition **(a)** from part (ii) holds but not condition **(b)**, which are *not* linearly dependent. (Hint: you can use without proof the fact that the 'bump function' $b(x)$, defined to be equal to $e^{-1/(1-x^2)}$ for $|x| < 1$ and zero otherwise, is infinitely differentiable for all $x \in \mathbb{R}$.)

Q2 Consider the following solution of the sine-Gordon equation:

$$u(x,t) = 4\arctan\left(t\,\mathrm{sech}(x)\right)\ .$$

2.1 Show that this solution contains a kink and an anti-kink.

2.2 Find the approximate trajectories $x_{\mathrm{k}}(t)$ and $x_{\mathrm{ak}}(t)$ of the centres of the kink (k) and the antikink (ak) at early times (large and negative t) and late times (large and positive t). Sketch these trajectories in the (x,t) plane. Finally, calculate the velocity $v_{\mathrm{k/ak}}(t) = \dot{x}_{\mathrm{k/ak}}(t)$ and the acceleration $a_{\mathrm{k/ak}}(t) = \ddot{x}_{\mathrm{k/ak}}(t)$ of the kink and the anti-kink at early and late times, and specify their signs.

2.3 Calculate the energy

$$E = \int_{-\infty}^{+\infty} dx\ \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + 1 - \cos(u)\right]$$

of a single static kink or anti-kink, which is described by the field

$$u(x,t) = 4\arctan\left(\exp(\pm(x - x_0))\right)\ .$$

You may use without proof the integral

$$\int_{-\infty}^{+\infty} dx\ \mathrm{sech}^2(x) = 2\ .$$

This energy E is the mass M of the kink or anti-kink.

2.4 Use the previous results and Newton's law $F = Ma$ (force = mass $\times$ acceleration) to find how the force F between a kink and an anti-kink depends on the distance d between them, when the distance is large. Is this force attractive or repulsive?

Q3 Let

$$[D_t^m D_x^n(F,G)](x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n F(x,t)G(x',t')\Big|_{\substack{x'=x\\ t'=t}} ,$$

where m, n are non-negative integers and F, G are any functions of x and t.

It is known that, if a pair of functions $f(x,t)$ and $g(x,t)$ obey the system of equations

$$\begin{cases} (D_x^3 + D_t)(f,g) = 0 \\ D_x^2(f,f) + D_x^2(g,g) = 0 \ , \end{cases}$$

then the field $u(x,t)$ given by

$$u = 2\frac{\partial}{\partial x}\arctan(g/f)$$

is a solution of the mKdV equation $u_t + 6u^2u_x + u_{xxx} = 0$.

3.1 Show that

$$u = 2\frac{D_x(g,f)}{g^2+f^2} \ .$$

3.2 Now assume that f and g take the form

$$f(x,t) = 1 + \epsilon \exp[\theta(x,t)] \ , \qquad g = 1 + \epsilon \exp[\hat{\theta}(x,t)] \ ,$$

where ϵ is a formal expansion parameter and

$$\theta(x,t) = ax + bt + c \ , \qquad \hat{\theta}(x,t) = \hat{a}x + \hat{b}t + \hat{c} \ ,$$

with constants $a, b, c, \hat{a}, \hat{b}, \hat{c}$. Working order by order in ϵ, find a real solution of the system of equations which depends on both x and t.

3.3 Find the corresponding solution $u(x,t)$ of the mKdV equation.

Q4 **4.1** Explain, in terms of their actions on other functions, what it means for two differential operators to be equal, and what it means for a differential operator to be multiplicative. If $D = d/dx$ and $g(x)$ is a general function of x, show that $Dg = gD + g_x$ as differential operators, and derive a formula expressing $D^2 g$ as a sum of terms in which all powers of D appear on the right.

4.2 Let $L = D^2 + u(x)$, with $u(x)$ some given function, and let $B = \alpha(x)D + \beta(x)$. Giving full details of your calculations, find the most general forms of the functions $\alpha(x)$ and $\beta(x)$ such that $[L, B]$ is multiplicative. If u also depends on t and $L_t + [L, B] = 0$, what partial differential equation must u satisfy?

4.3 Now let $M = x^2 D^2 + xD + u(x)$, with $u(x)$ some given function, and $C = \gamma(x)D$. Find the most general form of $\gamma(x)$ such that $[M, C]$ is multiplicative, and write down the partial differential equation for u that would follow from setting $M_t + [M, C]$ equal to zero.

4.4 Show that $x^2 D^2 + xD$ is self-adjoint with respect to the modified inner product

$$\langle f, g\rangle = \int_0^\infty f(x)^* g(x) \frac{dx}{x},$$

when acting on functions on $(0, \infty)$ for which $\langle f, f\rangle$ is finite.

Q5 For this question you can assume that situations such as found in part (iii) of question **1.4** do not arise, so that any two functions for which $W[f,g]$ vanishes will be linearly dependent.
Consider the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = k^2\psi(x)$$

where $V(x) \to 0$ as $x \to \pm\infty$.

5.1 Show that any two bound state eigenfunctions $\psi_1(x)$ and $\psi_2(x)$ sharing the same bound state eigenvalue $k^2 < 0$ must be linearly dependent. (Hint: first show that their Wronskian is constant.)

5.2 If $V(x)$ is symmetric, so that $V(x) = V(-x)$, show that all bound state eigenfunctions ψ are either even ($\psi(-x) = \psi(x)$) or odd ($\psi(-x) = -\psi(x)$). (Hint: first use the result of **5.1** to show that $\psi(x) \propto \psi(-x)$, and then consider the possible values of the proportionality constant.)

5.3 Now suppose that the potential $V(x)$ from **5.2** is given by

$$V(x) = -a\,\delta(x+r) - b\,\delta(x) - a\,\delta(x-r)$$

where a, b and r are real numbers with $r > 0$, and $\delta(x)$ is the Dirac delta function. Setting $k = i\mu$, $\mu > 0$ and normalising your solutions such that $\psi(x) \sim e^{-\mu x}$ as $x \to +\infty$, write down the general form that even and odd bound state eigenfunctions must take in each of the regions $x < -r$, $-r < x < 0$, $0 < x < r$ and $x > r$, and the matching conditions that should be imposed at $x = -r$, $x = 0$ and $x = r$. In each case your solutions should depend on just two undetermined parameters A and B, say, which you can take to be the coefficients of the two exponentials in the region $0 < x < r$.

5.4 Apply the matching conditions to the odd bound states, and eliminate A and B to find a constraint on μ that should be independent of b. Using a graphical method, show that this constraint has no solutions with $\mu > 0$ for $a \leq 0$, and one solution with $\mu > 0$ for $a > 0$ if (and only if) a is larger than a function of r that you should find.

5.5 Repeat the analysis of **5.4** to find a constraint on the even bound states, and analyse how the number of bound states depends on a and b when both are positive. You can assume that there are never more than two even bound states.