Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3281-WE01

Title:
Topology III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** For each part give a finite set and a topology on it that has the given properties. (There is no need to justify that it has the properties, just give the topology).

(i) Not connected and not Hausdorff.

(ii) Connected but not Hausdorff.

(iii) Hausdorff but not connected.

(iv) Both Hausdorff and connected.

1.2 We write M_2 for the two holed torus, the closed surface given by adding two handles to a sphere. By drawing a suitable graph on M_2, compute the Euler characteristic $\chi(M_2)$. State what properties your graph must have to make it suitable.

1.3 Let $\mathbb{C}^*$ denote the non-zero complex numbers and for $r > 0$ let C_r denote the circle in $\mathbb{C}^*$ of complex numbers of modulus r considered as a loop described in the anticlockwise direction. Let $f\colon \mathbb{C}^* \to \mathbb{C}$ be the function given by $f(z) = z^2 + \frac{1}{z}$ and let γ_r be the loop $f(C_r)$ in $\mathbb{C}$. For what values of r is the winding number of γ_r about the origin defined? Give two values of r for which the winding numbers of γ_r are defined but different, stating the values of the winding numbers for your examples. You should briefly justify your answer stating any results you use from lectures, but you do not need to do any detailed computations.

Q2 Consider the set

$$X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, (x, y) \neq (0, 0)\},$$

that is, the closed unit disc with the origin removed. We write τ_{E} for the topology on X induced from the usual Euclidean topology on $\mathbb{R}^2$.

For $p \in S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $0 \leq \lambda < \mu \leq 1$, let

$$L_{p,\lambda,\mu} = \{tp : \lambda < t < \mu\} \subset X$$

be the open straight-line segment between λp and μp.

2.1 Show that

$$\mathcal{B} = \tau_{\mathrm{E}} \cup \left(\bigcup_{\substack{p \in S^1 \\ 0 \leq \lambda < \mu \leq 1}} \{L_{p,\lambda,\mu}\} \right)$$

is the basis for a topology on X. We shall call this topology τ.

2.2 Determine whether or not (X, τ) is compact.

2.3 Show that (X, τ) is path-connected.

2.4 For each $x \in X$, determine the connected components of the space $X \setminus \{x\}$ with the subspace topology induced from τ.

Q3 Let

$$\tau = \{\mathbb{R} \setminus C : C \subset \mathbb{R} \text{ is compact with respect to the subspace topology on } C \text{ induced from the Euclidean topology on } \mathbb{R}\} \cup \{\emptyset\}.$$

3.1 Show that τ is a topology on $\mathbb{R}$.

3.2 Give an example of a topological space Y and a function $f\colon \mathbb{R} \to Y$ that is continuous with respect to exactly one of τ and the Euclidean topology.

3.3 Determine whether or not $(\mathbb{R}, \tau)$ is Hausdorff.

3.4 Determine whether or not $(\mathbb{R}, \tau)$ is compact.

3.5 Determine whether or not there exists a non-constant continuous function from $(\mathbb{R}, \tau)$ to $\mathbb{R}$ with the Euclidean topology.

Q4 **4.1** Let K be a finite simplicial complex that triangulates a closed surface S. Suppose $s\colon K \to K$ is a simplicial map such that $s \circ s = Id_K$, the identity map on K. The quotient space $L = K/s$ is defined by the equivalence relation $x \sim s(x)$ for all $x \in K$. Suppose that L has a simplicial complex structure such that the quotient map $q\colon K \to L$ is simplicial and one-to-one on any individual simplex in K. You are told that L also triangulates a closed surface R. We define a *fixed point* of s to be a point $x \in K$ such that $s(x) = x$.

(i) Suppose s has only a finite number of fixed points. Prove that all these fixed points must be 0-simplices of K.

(ii) Now suppose s has exactly a fixed points (and that all these are 0-simplices). Find and prove a relationship between the numbers

$$a,\ \chi(S) \text{ and } \chi(R)\,.$$

4.2 For closed surfaces S, R of your choice with $\chi(S) < \chi(R) < 0$, describe a homeomorphism $h\colon S \to S$ such that $S/h = R$ and use this to illustrate your answer to the first part of the question. (You do not need to give complete triangulations of your surfaces, but should indicate the fixed points, if there are any.)

Q5 **5.1** Is the open unit 3-ball E^3 homotopy equivalent to the closed unit 3-ball D^3? Justify your answer. You may state without proof any result from the lecture notes.

5.2 Let N be the closed surface given by taking the connected sum of a Klein bottle and a projective plane. Using any method you wish from the lectures, compute $\pi_1(N)$ explaining your calculations. You may assume knowledge of the fundamental group of S^1 and of any contractible space, but you should present as part of your answer the calculation of the fundamental group of any other space you use.