Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3291-WE01

Title: Partial Differential Equations III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Let $n \geq 2$ be an integer. For a four times differentiable function $u : \mathbb{R}^n \to \mathbb{R}$ we define the differential operator

$$\Delta^2 u := \Delta(\Delta u) = \sum_{i,j=1}^{n} \partial_{x_i}\partial_{x_i}\partial_{x_j}\partial_{x_j} u.$$

We aim to find the fundamental solution of Δ^2 on $\mathbb{R}^n$. For this, we consider the following questions.

1.1 Find all polynomial functions $P : \mathbb{R}^n \to \mathbb{R}$ such that $\Delta^2 P = 0$ everywhere on $\mathbb{R}^n$.

1.2 Now let $n = 2$. Find all radial functions $R : \mathbb{R}^2 \to \mathbb{R}$ (i.e. $R(x) = h(|x|)$, for some $h : [0, +\infty) \to \mathbb{R}$) such that $\Delta^2 R(x) = 0$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.

Hint: first, observe the crucial fact that $\Delta^2 = -\Delta \circ (-\Delta)$. Write down the corresponding ODE satisfied by h and rely on the ideas from the derivation of the fundamental solution of $-\Delta$ on $\mathbb{R}^2$ developed in the lectures.

1.3 Take the principal part of the solutions obtained in Question **1.2** (i.e. simply neglect the terms that fit in Question **1.1**). Use the condition $\int_{B_r(0)} \Delta^2 \Psi d\boldsymbol{x} = 1$ (for all $r > 0$, in a generalised sense) to determine all the necessary constants and find the fundamental solution Ψ on $\mathbb{R}^2$. Here, $B_r(0)$ stands for the standard open ball in $\mathbb{R}^2$ with radius $r > 0$ around 0.

Hint: use the divergence theorem.

Q2 We consider the Cauchy problem associated to Burgers' equation, i.e.

$$\begin{cases} \partial_t u + \partial_x(u^2/2) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Let $u_0 : \mathbb{R} \to \mathbb{R}$ be given by

$$u_0(x) = \begin{cases} 3, & x < -1, \\ 0, & -1 \leq x < 0, \\ 6, & 0 \leq x < 1, \\ 0, & 1 \leq x. \end{cases}$$

Find a possibly unbounded entropy solution (i.e. a weak integral solution that satisfies the one sided jump conditions) to this problem. To achieve this, consider the following steps.

2.1 Find the characteristic lines associated to the problem and sketch them.

2.2 Introduce possible shock curves that satisfy the Rankine-Hugoniot condition. If there is a need for 'void filling', introduce either possible new shocks or rarefaction waves.

2.3 Write down the candidate for the global entropy solution (we allow this to be unbounded; you do not need to verify that this satisfies the definition of the entropy solution).

Q3 Let $\Omega \subseteq \mathbb{R}^2$ be a given domain with smooth boundary and let $V : \Omega \to \mathbb{R}$ be a given smooth function. Furthermore, we consider $\Gamma := \partial\Omega$ (the boundary of Ω) and $g : \Gamma \to \mathbb{R}$ to be a given smooth function. For the unknown function $u : \Omega \to \mathbb{R}$, we consider the boundary value problem

$$\begin{cases} \frac{1}{2}|\nabla u(\boldsymbol{x})|^2 = V(\boldsymbol{x}), & \boldsymbol{x} = (x_1, x_2) \in \Omega, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma. \end{cases} \tag{1}$$

Here we use the notation $|\boldsymbol{a}| := \sqrt{a_1^2 + a_2^2}$, for any $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^2$.

3.1 With respect to linearity (linear, semi-linear, etc.), what is the type of the PDE in (1)? Justify your answer!

3.2 If $V(\boldsymbol{x}) = -\exp(-|\boldsymbol{x}|^2)$ and g is arbitrary, show that the problem (1) does not have a classical solution.

For the following questions, let us set $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\}$, $V(\boldsymbol{x}) = \frac{1}{2}|x|^2$ and $g(\boldsymbol{x}) = 0$. We aim to find a solution to (1) by the method of characteristics. For this, use the parameter $s \in \mathbb{R}$ to give a parametrisation of Γ. Denote the characteristics by $\boldsymbol{X}(\tau, s) = (X_1(\tau, s), X_2(\tau, s))$, $(\tau \geq 0, s \in \mathbb{R})$ and the solution along the characteristics by $z(\tau, s) = u(\boldsymbol{X}(\tau, s))$, $(\tau \geq 0, s \in \mathbb{R})$.

3.3 By the nature of the problem, we need to introduce a new variable $\boldsymbol{P}(\tau, s) = (P_1(\tau, s), P_2(\tau, s))$, $(\tau \geq 0, s \in \mathbb{R})$ by setting

$$\boldsymbol{P}(\tau, s) = \nabla u(\boldsymbol{X}(\tau, s)), \ \tau > 0, \ s \in \mathbb{R}.$$

Find an ODE system satisfied by $(\boldsymbol{X}(\tau, s), \boldsymbol{P}(\tau, s), z(\tau, s))$ together with the corresponding initial conditions. *Hint:* compute $\partial_\tau \boldsymbol{P}$ and differentiate the PDE in (1) with respect to x_1 and x_2. One of the equations should be $\partial_\tau \boldsymbol{X} = \boldsymbol{P}$.

3.4 Solve the ODE system that you have found in Question **3.3**.

3.5 By inverting the flow, find a local solution to the PDE in (1) in a neighbourhood of Γ. Determine this neighbourhood as well.

3.6 Is the solution that you have found in Question **3.5** global? Can you find other solutions? Justify your answers!

Q4 Let $n \geq 2$ be a given integer and let $\Omega \subset \mathbb{R}^n$ be a given bounded, connected open set with smooth boundary. Let moreover $f : \mathbb{R} \to \mathbb{R}$ be a given continuous function such that $|f(s)| \leq C_f|s|$ for all $s \in \mathbb{R}$, for some $C_f > 0$. We are interested in the solutions $u : \Omega \to \mathbb{R}$ to the boundary value problem

$$\begin{cases} \Delta^2 u = f(u), & \text{in } \Omega, \\ u = \partial_{x_i} u = 0,\ i \in \{1, \dots, n\}, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

For a four times differentiable function $u : \Omega \to \mathbb{R}$ we use the notation

$$\Delta^2 u := \Delta(\Delta u) = \sum_{i,j=1}^{n} \partial_{x_i}\partial_{x_i}\partial_{x_j}\partial_{x_j} u.$$

4.1 Show that there exists a constant $C = C(n, \Omega) > 0$ depending only on Ω and n, such that for any $u \in C^2(\overline{\Omega})$ with $u = \partial_{x_i} u = 0,$ on $\partial\Omega,\ i \in \{1, \dots, n\}$, we have

$$\|\nabla u\|_{L^2(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)}.$$

4.2 Deduce from Question **4.1** that there exists a constant $\tilde{C} = \tilde{C}(n, \Omega)$ depending only on Ω and n, such that for any $u \in C^2(\overline{\Omega})$ with $u = \partial_{x_i} u = 0,$ on $\partial\Omega,\ i \in \{1, \dots, n\}$, we have

$$\|u\|_{L^2(\Omega)} \leq \tilde{C}\|\Delta u\|_{L^2(\Omega)}.$$

4.3 Show that the constant function 0 is always a classical solution to the problem (2).

4.4 We say that $u \in C^2(\overline{\Omega})$ is a weak solution to (2), if

$$\int_\Omega (\Delta u)(\Delta\varphi)\mathrm{d}\boldsymbol{x} = \int_\Omega f(u)\varphi\mathrm{d}\boldsymbol{x},$$

for all $\varphi \in C^2(\overline{\Omega})$, with $\varphi = \partial_{x_i}\varphi = 0$ on $\partial\Omega,\ i \in \{1, \dots, n\}$. Give a sufficient condition involving n, Ω and C_f that ensures that the problem (2) does not have a nonzero weak solution in the class $C^2(\overline{\Omega})$.

Hint: use a specific test function in the weak formulation, then use the previously established inequalities and detect a condition which leads to a contradiction.

Q5 Let $n \geq 2$ be a given integer, let $\Omega \subset \mathbb{R}^n$ be a given bounded connected open set with smooth boundary $\partial\Omega$. Let moreover $g : \partial\Omega \to \mathbb{R}$ be a given continuous function. We consider the solution to the boundary value problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Recall that $u \in C^1(\overline{\Omega})$ is said to be a weak solution to (3) if $u = g$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi \mathrm{d}\boldsymbol{x} = 0,$$

for all $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$.

5.1 Show that (3) has at most one weak solution.

5.2 Determine the energy functional and the minimisation problem for which the weak formulation of (3) can be seen as a first order necessary optimality condition.

5.3 Fix $\boldsymbol{x}_0 \in \Omega$ and $r, R > 0$ such that $B_r(\boldsymbol{x}_0) \subset B_R(\boldsymbol{x}_0) \subset \Omega$, where $B_\rho(\boldsymbol{y})$ denotes the open ball of radius $\rho > 0$ centered at $\boldsymbol{y}$. Suppose that $u \in C^1(\overline{\Omega})$ is a weak solution to (3). Show that there exists a constant $C > 0$ such that

$$\int_{B_r(x_0)} |\nabla u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \leq \frac{C}{(R-r)^2} \int_{B_R(x_0) \setminus B_r(x_0)} |u(\boldsymbol{x}) - \lambda|^2 \mathrm{d}\boldsymbol{x}, \ \forall \lambda \in \mathbb{R}.$$

Hint: consider a special test function of the form $\varphi := (u - \lambda)\eta^2$ and plug it into the weak formulation satisfied by u. Here choose $\eta \in C_c^1(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(\boldsymbol{x}_0)$ and $\eta \equiv 0$ on $\Omega \setminus B_R(\boldsymbol{x}_0)$. What property does $\nabla \eta$ need to satisfy? You do not need to construct such an η explicitly.