Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3301-WE01

Title: Mathematical Finance III

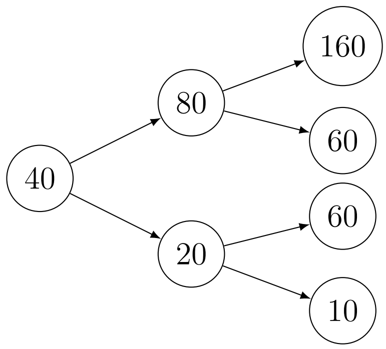
Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Consider the following 2-period Binomial market shown as a tree.



The interest rate is $3/4$. Answer the following questions and show your work.

1.1 Calculate the martingale probabilities along the edges of the tree.

1.2 Consider the contingent claim $X = S_0 + S_1 - S_2$. Calculate the arbitrage-free price of X at time 0.

1.3 Explain in words the meaning or significance of the price of X from part **1.2**.

Let $(W_t, t \geq 0)$ be a Brownian motion started at $W_0 = 0$.

1.4 Use Itô's formula to show that if $h : \mathbb{R} \to \mathbb{R}$ is differentiable, then

$$\int_0^t h(s)\,\mathrm{d}W_s = h(t)W_t - \int_0^t h'(s)W_s\,\mathrm{d}s, \quad (t \geq 0).$$

Define $G_0 := 0$, and, for $t > 0$, $G_t := t^{-1}\int_0^t W_s\,\mathrm{d}s$. Also let $Y_t := W_t - G_t$ for $t \geq 0$.

1.5 Show that $G_t \sim \mathcal{N}(0, \sigma_t^2)$ and $Y_t \sim \mathcal{N}(0, \sigma_t^2)$, where σ_t^2 is a function of $t \geq 0$ that you should determine.

1.6 Find $\mathbb{C}\mathrm{ov}(G_t, Y_t)$.

1.7 Do $(G_t, t \geq 0)$ and $(Y_t, t \geq 0)$ have the same distribution as processes? Explain briefly.

Q2 Consider a market consisting of two periods $t = 0$ and T, a stock S_t and interest compounded continuously at rate r. Assume the market has no arbitrage. There are two contingent claims X and Y as follows.

Claim X: At time $t = T$ the holder of the claim has the right, but not the obligation, to buy 1 unit of the stock for a strike price K. If the holder does not exercise the option to buy the stock, then the holder has to pay a penalty of price J.

Claim Y: At time $t = T$ the holder of the claim has the right, but not the obligation, to sell 1 unit of the stock for a strike price K. If the holder does not exercise the option to sell the stock, then the holder instead gains an amount J.

2.1 Show, with proper explanation, that the value of X at time T is $\max\{S_T - K, -J\}$ and the value of Y at time T is $\max\{K - S_T, J\}$.

2.2 Let C be the price of X at $t = 0$ and P be the price of Y at $t = 0$. Prove that $P + S_0 = C + (K + J)\mathrm{e}^{-rT}$.

2.3 Prove or disprove whether the following inequalities hold for every value of J and K:

$$\text{(i) } P \leq S_0; \qquad \text{(ii) } C \leq S_0.$$

You must explain your reasoning.

Q3 Consider a market $\mathcal{M} = (B_t, S_t)$ consisting of a bond and a stock as follows for $t = 0, T$.

$$B_0 = 1, \quad B_T = 4/3.$$

$$S_0 = 10, \quad S_T = \begin{cases} 20 & \text{with probability } 1/3, \\ 15 & \text{with probability } 1/3, \\ 5 & \text{with probability } 1/3. \end{cases}$$

A portfolio in this market is a vector $h = (x, y) \in \mathbb{R}^2$. The value of the portfolio h is $V_t^h = xB_t + yS_t$ for $t = 0, T$.

3.1 The market contains arbitrage if there is a non-zero portfolio h such that $V_0^h = 0$, $V_T^h \geq 0$ with probability 1 and $V_T^h > 0$ with positive probability. Prove that this market contains no arbitrage.

3.2 A portfolio h is replicating for a contingent claim X if $V_T^h = X$ almost surely. Consider the following two contingent claims X_1 and X_2.

$$X_1 = \begin{cases} 12 & \text{if } S_T = 20 \\ 10 & \text{if } S_T = 15 \\ 6 & \text{if } S_T = 5 \end{cases} \qquad X_2 = \begin{cases} 8 & \text{if } S_T = 20 \\ 3 & \text{if } S_T = 15 \\ 1 & \text{if } S_T = 5 \end{cases}$$

For each of X_1 and X_2, either find a replicating portfolio or prove that none exists. Show your work neatly.

3.3 Consider the contingent claim X_1 from part **3.2**. Find the arbitrage-free price of X_1. Justify your answer.

3.4 A martingale measure for this market is a change of measure $\mathbb{Q}$ on the stock S_T such that under $\mathbb{Q}$ the following identity holds:

$$\frac{1}{B_T}\mathbb{E}_{\mathbb{Q}}[S_T] = S_0.$$

Find, with proper justification, all martingale measures in this market.

Q4 Suppose that $(X_t, t \geq 0)$ is an Itô process with $X_0 = x_0 > 0$, $X_t \geq 0$ for all $t \geq 0$, satisfying the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \left[2\sqrt{X_t} - \mu X_t + \sigma^2\right] \mathrm{d}t + 2\sigma\sqrt{X_t}\mathrm{d}W_t, \quad t \geq 0,$$

where $(W_t, t \geq 0)$ is a Brownian motion and $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants.

4.1 For $\alpha \in \mathbb{R}$, let $Y_t = \mathrm{e}^{\mu t/2} X_t^\alpha$. Derive an SDE for Y_t (you may leave the right-hand side in terms of X_t). There is a special value $\alpha = \alpha_\star$ for which the drift in the SDE for Y_t is deterministic. What is $\alpha_\star$?

4.2 Solve the SDE for Y_t that you obtained in question **4.1** in the special case $\alpha = \alpha_\star$. Hence find an expression for $X_t^{\alpha_\star}$.

4.3 Calculate $\mathbb{E}(X_t^{\alpha_\star})$ and $\mathbb{V}\mathrm{ar}(X_t^{\alpha_\star})$. Carefully justify your calculations.

4.4 Find $\lim_{t\to\infty} \mathbb{E}(X_t)$; you should consider the cases $\mu < 0$, $\mu = 0$, and $\mu > 0$.

Q5 Consider the continuous-time Black–Scholes market, with price dynamics given by

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t, \qquad \mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$

where $r > 0$ is the risk-free interest rate, μ and σ are constant parameters, and $(W_t, t \geq 0)$ is a Brownian motion under the real-world measure $\mathbb{P}$.

A contingent claim X_T with expiry time T and threshold $K > 0$ is given by

$$X_T = \mathbb{1}\left\{\max_{0\leq t\leq T} S_t \geq K\right\}.$$

5.1 Show that the arbitrage-free price $\Pi_0(X_T)$ at time 0 of X_T can be expressed in terms of an expectation under the risk-neutral measure $\mathbb{Q}$ satisfying

$$\mathbb{E}_{\mathbb{Q}}[X_T] = H(T, \alpha, y),$$

where α, y are functions of S_0, K, σ, and r, and

$$H(T, \alpha, y) = \mathbb{P}\left(\max_{0\leq t\leq T}\left(\alpha t + W_t\right) \geq y\right). \tag{1}$$

5.2 By considering a change of measure under which $(\alpha t + W_t, t \geq 0)$ is a Brownian motion, show that the probability in (1) can be written in terms of

$$\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\alpha W_T}\,\mathbb{1}\{M_T \geq y\}\right], \text{ where } M_T := \max_{0\leq t\leq T} W_t. \tag{2}$$

Give a careful explanation of the application of any theorem from lectures that you use.

5.3 Take $T = 1$. It can be shown that the joint distribution of W_1 and M_1 is given by

$$\mathbb{P}\left(M_1 \geq y,\, W_1 \in [x, x + \mathrm{d}x]\right) = \begin{cases} \phi(x)\,\mathrm{d}x & \text{if } x \geq y, \\ \phi(2y - x)\,\mathrm{d}x & \text{if } x < y, \end{cases} \tag{3}$$

where ϕ is the standard normal density function. Use (3) to compute (2) and hence determine $\Pi_0(X_1)$. Your answer may be written in terms of $N(x)$, the cumulative distribution function of the standard normal distribution.

5.4 Give a brief explanation of the origin of formula (3).