

## EXAMINATION PAPER

Examination Session: May/June

2021

Year:

Exam Code:

MATH3351-WE01

## Title:

## Statistical Mechanics III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.
	Please start each question on a new page. Please write your CIS username at the top of each page.
	To receive credit, your answers must show your working and explain your reasoning.

Revision:



**Q1** Here we study a random walk in two dimensions with coordinates (x, y). Consider a particle taking a series of random steps; each step has equal length a and its direction is chosen at random. The particle starts at the origin: after N steps, its position is a random variable:

$$x_N = a \sum_{i=1}^N \cos \theta_i$$
  $y_N = a \sum_{i=1}^N \sin \theta_i$ 

where each of the  $\theta_i$  is an independent random variable with a uniform distribution over  $[0, 2\pi)$ .

- **1.1** Calculate the expectation value of the position  $(\langle x_N \rangle, \langle y_N \rangle)$  after N steps.
- 1.2 Calculate the square root of the expectation value of the squared displacement from the origin  $\sqrt{\langle x_N^2 + y_N^2 \rangle}$ .

Consider a system of N distinguishable particles, at temperature T. Each particle can be in a microstate corresponding to energy  $E_0 = 0$  or  $E_1 = \varepsilon > 0$ . The lower energy level has  $g_0$  different microstates while the higher energy one has  $g_1$  different microstates.

- **1.3** Compute the partition function and the average energy  $\langle E \rangle$  of the total system.
- **1.4** Compute the heat capacity  $C_{V,N}$  and from that infer the variance  $\sigma_E^2$  of the energy.
- ${\bf Q2}$  Consider a reversible thermodynamic cycle which consists of the following 4 processes:
  - AB: adiabatic compression from  $A = (V_1, p_1)$  to  $B = (V_2, p_2)$ .
  - BC: volume increase at constant pressure  $p_2$  from  $B = (V_2, p_2)$  to  $C = (V_3, p_2)$ .
  - CD: adiabatic expansion from  $C = (V_3, p_2)$  to  $D = (V_4, p_1)$ .
  - DA: volume decrease at constant pressure  $p_1$  from  $D = (V_4, p_1)$  to  $A = (V_1, p_1)$ .

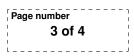
Take the working substance of the system to be an ideal gas with the equation of state  $pV = Nk_BT$  and internal energy  $E = \alpha Nk_BT$ , with  $\alpha$  a constant.

- **2.1** Draw the cycle in the (p, V) plane. Calculate the work done on the system and the heat absorbed in each of the four processes.
- **2.2** Let W be the total work done on the system in a cycle and let  $Q_{BC}$  be the heat absorbed by the system in the process BC. Compute the efficiency of the engine:

$$\eta = \left| \frac{W}{Q_{BC}} \right|$$

in terms of the temperatures  $T_{A/B/C/D}$  of the states A, B, C, and D. Compare your result to that of a Carnot engine and comment on the discrepancy, if any.

- **2.3** Express your answer for the efficiency in terms of the ratio of pressures  $p_1/p_2$ .
- **2.4** Compute the change in entropy of the system along each of the legs of the cycle.



- Exam code MATH3351-WE01
- **Q3** In this problem we will consider the following Hamiltonian for a classical particle living in one dimension under the influence of gravity:

$$H(q,p) = \frac{p^2}{2m} + mgq$$

where the motion of the particle q is restricted to q > 0, and the constants m, g are both positive.

Parts 3.1 and 3.2 may be attempted in either order.

**3.1** (i) Write down Hamilton's equations for the motion of the particle. Consider the initial conditions:

$$q(t=0) = q_0$$
  $p(t=0) = 0$ 

where  $q_0$  is a positive constant. Find the solution to the equations of motion satisfying the above initial conditions for t > 0 and for  $t < t^*$ , where  $t^*$  is defined as the time when  $q(t = t^*) = 0$ .

(ii) Consider the following normalized probability density function for the particle at a time t = 0:

$$\rho_a(q, p, t = 0) = \delta(q - q_0)\delta(p)$$

Solve for the time evolution of this probability density at later times  $\rho_a(q, p, t)$  with  $0 < t < t^*$ , where  $t^*$  is defined as above.

- **3.2** (i) Draw the contours of constant energy E = H(q, p) in phase space (q, p) for the particle.
  - (ii) In the microcanonical ensemble, compute the "area" of accessible states  $\mathcal{N}(E)$ , defined as

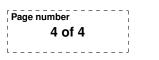
$$\mathcal{N}(E) = \int_{\mathcal{P}} dq dp \,\,\delta(H(q, p) - E)$$

- (iii) Compute the entropy S(E) and the temperature T(E) of this particle in the microcanonical ensemble. (You may take the number of states  $\Omega(E) = \mathcal{N}(E)$ ). Calculate the mean position  $\langle q(E) \rangle$  of the particle in the microcanonical ensemble.
- Q4 Consider a classical gas of N indistinguishable and non interacting particles which is placed inside a volume V and held at temperature T. The Hamiltonian of a single particle can be written as,

$$H = a \left( p_x^2 + p_y^2 + p_z^2 \right)^{b/2}, \quad a, b > 0.$$

- 4.1 Compute the single particle partition function and from that infer the partition function of the system of N particles.
- **4.2** Compute the internal energy  $\langle E \rangle$  and the pressure *P*.
- **4.3** Consider the thermodynamic limit, and compute the chemical potential  $\mu$ .

**Hint:** At some point you should find useful the Gamma function defined through  $\Gamma(\gamma) = \int_0^{+\infty} x^{\gamma-1} e^{-x} dx.$ 





**Q5** You are given a finite temperature system of two identical particles each of which may occupy any of the three energy levels,

$$E_n = n \varepsilon, \quad n = 0, 1, 2.$$

Moreover, the lowest energy microstate with  $E_0 = 0$  is doubly degenerate while the levels for n = 1, 2 are non-degenerate. For each of the following cases enumerate the inequivalent configurations, determine the partition function and the average energy.

- 5.1 The particles obey Fermi statistics
- 5.2 The particles obey Bose statistics
- 5.3 The particles are distinguishable
- 5.4 Without performing any calculation, which of the above systems would have the smallest entropy at zero temperature? Justify your answer.