Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH3391-WE01

Title:
Quantum Computing III

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

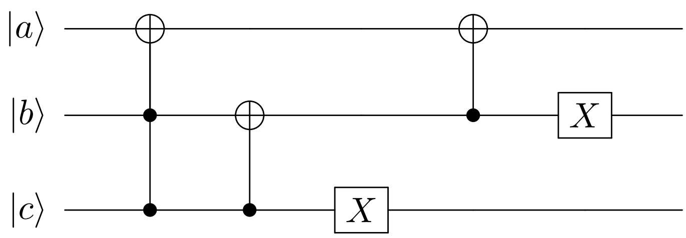
Q1 **1.1** The "square-root of NOT". Let us consider the quantum gate acting on a single qubit defined by

$$\sqrt{\text{NOT}}\,|0\rangle = \frac{(1+i)\,|0\rangle + (1-i)\,|1\rangle}{2}\,,$$
$$\sqrt{\text{NOT}}\,|1\rangle = \frac{(1-i)\,|0\rangle + (1+i)\,|1\rangle}{2}\,.$$

Why is this gate called $\sqrt{\text{NOT}}$?

1.2 Using the quantum universal gate set $\{CNOT, H, T\}$ construct a circuit to implement the $\sqrt{\text{NOT}}$ gate defined in part **1.1**.

1.3 The action of the following quantum circuit on the computational basis states can be interpreted in terms of addition modulo 8. Explain precisely what this circuit does in terms of this interpretation, and explain why (without just listing the outputs for all possible computational basis state inputs).



Give a simple generalisation of the circuit which will implement the same addition, not modulo 8. (I.e. if the first circuit has output $(x+y) \mod 8$ then your circuit should have output $x+y$, with the same identification of x and y.) You can use any of the standard single-qubit gates $\{T, S, H, X, Y, Z\}$ as well as controlled NOT gates with any number of controls.

Q2 Remember that the trace-distance $D(\hat{\rho}_1, \hat{\rho}_2)$ between two arbitrary states $\hat{\rho}_1$ and $\hat{\rho}_2$ is given by

$$D(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{2}\mathrm{Tr}\Big(|\hat{\rho}_1 - \hat{\rho}_2|\Big)$$

2.1 Show that the trace-distance $D(\hat{\rho}_1, \hat{\rho}_2)$ is invariant under time-evolution.

2.2 You are now given $\hat{\rho}_1$ and $\hat{\rho}_2$, two generic mixed single-qubit states. Consider

$$D(\hat{U}\hat{\rho}_1\hat{U}^\dagger, \hat{\rho}_2)$$

the trace distance between $\hat{U}\hat{\rho}_1\hat{U}^\dagger$ and $\hat{\rho}_2$ as a function of the unitary transformation $\hat{U}$.

Find the minimum and maximum value of this trace distance as we vary $\hat{U}$. [Hint: Think geometrically]

2.3 The fidelity between a pure state $|\psi\rangle$ and a state $\hat{\rho}$ is given by

$$F(|\psi\rangle\,, \hat{\rho}) = \langle\psi|\,\hat{\rho}\,|\psi\rangle\ .$$

First compute the fidelity between $|\psi\rangle$, a generic pure single-qubit state, and $\hat{\rho}$, a generic mixed single-qubit state, and then find the minimum and maximum value of the fidelity between $\hat{U}\,|\psi\rangle$ and $\hat{\rho}$ as a function of the unitary transformation $\hat{U}$ where again $|\psi\rangle$ is a generic pure single-qubit state and $\hat{\rho}$ is a generic mixed single-qubit state.

Q3 "Super-dense comparison". Suppose we are given an unknown Boolean function f, i.e. a function $f : \{0,1\} \mapsto \{0,1\}$.

3.1 How many bits of information are needed to decide whether $f(0) = f(1)$ or $f(0) \neq f(1)$?

3.2 Define now a 2-qubit quantum gate given by

$$\hat{U}_f\,|a,b\rangle = |a, b\oplus f(a)\rangle\ ,$$

with $a, b \in \{0,1\}$ and $\oplus$ denotes the usual sum modulo 2, i.e.

$$a\oplus b = a + b\,(\mathrm{mod}\,2)\,,\ \text{i.e.}\qquad 0\oplus 0 = 0\,,\qquad 0\oplus 1 = 1\oplus 0 = 1\,,\qquad 1\oplus 1 = 0\,.$$

Evaluate $\hat{U}_f$ on the state $|+,-\rangle$ and then specialise your result to the four possible values for the pair $\{f(0), f(1)\}$. Remember that the states $|\pm\rangle$ are given by $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

3.3 Using the unitary gate $\hat{U}_f$ can you construct a protocol such that we can distinguish whether $f(0) = f(1)$ or $f(0) \neq f(1)$ with a single measurement, hence the "Super-dense comparison" in contrast to part **3.1**.

[HINT: Hadamard is your friend]

Q4 Consider an n-qubit Hilbert space with a k-dimensional subspace spanned by k computational basis states $\{|a_1\rangle, |a_2\rangle, \ldots, |a_k\rangle\}$ for some $k \in \{1, 2, \ldots, N-1\}$ where $N = 2^n$ and we do not know the elements of the set $A = \{a_1, a_2, \ldots, a_k\}$. We define the states

$$\begin{aligned} |a\rangle &\equiv \frac{1}{\sqrt{k}} \sum_{\alpha \in A} |\alpha\rangle \\ |a_\perp\rangle &\equiv \frac{1}{\sqrt{N-k}} \sum_{\alpha \notin A} |\alpha\rangle \\ |\phi\rangle &\equiv \frac{1}{\sqrt{N}} \sum_{\alpha=0}^{N-1} |\alpha\rangle \end{aligned}$$

where $\sum_{\alpha \notin A}$ denotes the sum over all integers in the interval $[0, N-1]$ which are not in the set A.

4.1 Write $|\phi\rangle$ in the form $|\phi\rangle = \cos\theta\, |a_\perp\rangle + \sin\theta\, |a\rangle$, giving a relation between θ, k and N.

4.2 Express the operators $M \equiv |a\rangle\langle a| + |\phi\rangle\langle\phi|$ and $U_\lambda \equiv \exp(i\lambda M)$ as 2×2 matrices when working in the 2-dimensional subspace with basis

$$\left\{ |a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |a_\perp\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} .$$

You may use the identity for the exponential of a linear combination of Pauli σ-matrices

$$\exp(i\boldsymbol{v} \cdot \boldsymbol{\sigma}) = \cos(v) I + i \sin(v) \hat{\boldsymbol{v}} \cdot \boldsymbol{\sigma}$$

where $v \equiv |\boldsymbol{v}|$ and $\boldsymbol{v} = v\hat{\boldsymbol{v}}$.

4.3 Assume that we implement $|\phi\rangle \to U_\lambda |\phi\rangle$ for any chosen $\lambda \in \mathbb{R}$ and then measure in the computational basis. What is the probability that the measurement outcome will be in the set A? What is the smallest positive value of λ to maximise this probability, and what is this maximum probability?

4.4 For $k = 1$, briefly comment on the similarities and differences between this process and Grover's algorithm.

Q5 Suppose we wish to encode a single logical qubit in an n-qubit Hilbert space using the code subspace given by $\{|\bar{0}\rangle = |00\cdots 0\rangle, |\bar{1}\rangle = |11\cdots 1\rangle\}$.

5.1 Calculate n_{min}, the lower bound on n in order to be able to correct for up to k bit-flips (each acting on a single qubit as $|0\rangle \leftrightarrow |1\rangle$) for an arbitrary state in the above code subspace. Show that for $k = 2$ the bound is $n \geq n_{min} = 5$.

5.2 Suppose we have a set of Hermitian operators which each square to the identity. What other property or properties must these operators satisfy in order for this set to be an error syndrome? How does the minimum number of such operators depend on n_{min}?

From now on we fix $k = 2$ and $n = 5$.

5.3 Find an error syndrome containing the minimum number of operators and including the operators $M_1 = Z_1 Z_0$ and $M_2 = Z_4 Z_3$ where Z_i denotes the operators acting as the Pauli matrix $Z = \sigma_3$ on qubit $i \in \{0, 1, 2, 3, 4\}$ and as the identity on the other qubits. (The subscripts on M_1, M_2 and σ_3 do not have this meaning.)

5.4 Explain how the error syndrome can be used to correct for arbitrary errors of up to 2 bit-flips, or any linear superposition of such errors. You do not need to explicitly detail every separate case but you should give enough detail to make it clear that such error correction is possible and include a general description of the method to both detect and correct errors. Give explicit details for the example of a state which has been transformed into a linear superposition of the original state $|\psi\rangle = \alpha |\bar{0}\rangle + \beta |\bar{1}\rangle$, the state $|\psi\rangle$ with a single bit-flip on qubit 0 and the state $|\psi\rangle$ with bit-flips on both qubits 0 and 1.