EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2021 | MATH4041-WE01 |

| Title: |
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| Partial Differential Equations IV |

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| Time (for guidance only): | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | Yes | Models Permitted: There is no restriction on the model of calculator which may be used. |

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| Instructions to Candidates: | Credit will be given for your answers to all questions.<br>All questions carry the same marks.<br><br>Please start each question on a new page.<br>Please write your CIS username at the top of each page.<br><br>To receive credit, your answers must show your working and explain your reasoning. |

| Revision: | |
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ED01/2021

**Q1** **1.1** If $f \in \mathcal{D}'(\mathbb{R})$ is a distribution and $\phi \in C^\infty(\mathbb{R})$ is a given function, show that $g := \phi f$ defined as

$$(g, \psi) = (\phi f, \psi) := (f, \phi\psi), \quad \forall \psi \in \mathcal{D}(\mathbb{R})$$

is a distribution.
*Hint:* you need to check whether $g$ is well-defined, linear and continuous.

**1.2** Recall that a distribution $f \in \mathcal{D}'(\mathbb{R})$ is zero if and only if

$$(f, \psi) = 0, \quad \forall \psi \in \mathcal{D}(\mathbb{R}).$$

Find all distributions $f$ such that

$$\phi f = 0, \tag{1}$$

i.e. the distribution $\phi f$ is the zero distribution, where $\phi : \mathbb{R} \to \mathbb{R}$ is given as $\phi(x) = e^x$.

We aim to find now all distributions $f$ such that (1) holds with the choice of

$$\phi(x) = (x - 1)^2.$$

**1.3** For this, show that it is enough to find $f$ such that

$$(f, \psi) = 0, \ \forall \psi \in \mathcal{D}(\mathbb{R}) \ \text{ with } \ \psi(1) = 0, \ \psi'(1) = 0.$$

*Hint:* you need to show which kind of test functions you can generate in the form of $(x-1)^2\Psi$, for arbitrary $\Psi \in \mathcal{D}(\mathbb{R})$.

**1.4** Relying on Question **1.3**, find a nonzero $f$ that satisfies (1) with $\phi(x) = (x-1)^2$.
*Hint:* try to construct special test functions $\psi$ that satisfy the conditions from Question **1.3**, using arbitrary test functions and some special ones.

**Q2** We consider the Cauchy problem associated to Burgers' equation, i.e.

$$\begin{cases} \partial_t u + \partial_x(u^2/2) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Let $u_0 : \mathbb{R} \to \mathbb{R}$ be given by

$$u_0(x) = \begin{cases} 3, & x < -1, \\ 0, & -1 \le x < 0, \\ 6, & 0 \le x < 1, \\ 0, & 1 \le x. \end{cases}$$

Find a possibly unbounded entropy solution (i.e. a weak integral solution that satisfies the one sided jump conditions) to this problem. To achieve this, consider the following steps.

**2.1** Find the characteristic lines associated to the problem and sketch them.

**2.2** Introduce possible shock curves that satisfy the Rankine-Hugoniot condition. If there is a need for 'void filling', introduce either possible new shocks or rarefaction waves.

**2.3** Write down the candidate for the global entropy solution (we allow this to be unbounded; you do not need to verify that this satisfies the definition of the entropy solution).

**Q3** Let $\Omega \subseteq \mathbb{R}^2$ be a given domain with smooth boundary and let $V : \Omega \to \mathbb{R}$ be a given smooth function. Furthermore, we consider $\Gamma := \partial\Omega$ (the boundary of $\Omega$) and $g : \Gamma \to \mathbb{R}$ to be a given smooth function. For the unknown function $u : \Omega \to \mathbb{R}$, we consider the boundary value problem

$$\begin{cases} \frac{1}{2}|\nabla u(\boldsymbol{x})|^2 = V(\boldsymbol{x}), & \boldsymbol{x} = (x_1, x_2) \in \Omega, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma. \end{cases} \tag{2}$$

Here we use the notation $|\boldsymbol{a}| := \sqrt{a_1^2 + a_2^2}$, for any $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^2$.

**3.1** With respect to linearity (linear, semi-linear, etc.), what is the type of the PDE in (2)? Justify your answer!

**3.2** If $V(\boldsymbol{x}) = -\exp(-|\boldsymbol{x}|^2)$ and $g$ is arbitrary, show that the problem (2) does not have a classical solution.

For the following questions, let us set $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\}$, $V(\boldsymbol{x}) = \frac{1}{2}|x|^2$ and $g(\boldsymbol{x}) = 0$. We aim to find a solution to (2) by the method of characteristics. For this, use the parameter $s \in \mathbb{R}$ to give a parametrisation of $\Gamma$. Denote the characteristics by $\boldsymbol{X}(\tau, s) = (X_1(\tau, s), X_2(\tau, s))$, $(\tau \geq 0, s \in \mathbb{R})$ and the solution along the characteristics by $z(\tau, s) = u(\boldsymbol{X}(\tau, s))$, $(\tau \geq 0, s \in \mathbb{R})$.

**3.3** By the nature of the problem, we need to introduce a new variable $\boldsymbol{P}(\tau, s) = (P_1(\tau, s), P_2(\tau, s))$, $(\tau \geq 0, s \in \mathbb{R})$ by setting

$$\boldsymbol{P}(\tau, s) = \nabla u(\boldsymbol{X}(\tau, s)), \ \tau > 0, \ s \in \mathbb{R}.$$

Find an ODE system satisfied by $(\boldsymbol{X}(\tau, s), \boldsymbol{P}(\tau, s), z(\tau, s))$ together with the corresponding initial conditions. *Hint:* compute $\partial_\tau \boldsymbol{P}$ and differentiate the PDE in (2) with respect to $x_1$ and $x_2$. One of the equations should be $\partial_\tau \boldsymbol{X} = \boldsymbol{P}$.

**3.4** Solve the ODE system that you have found in Question **3.3**.

**3.5** By inverting the flow, find a local solution to the PDE in (2) in a neighbourhood of $\Gamma$. Determine this neighbourhood as well.

**3.6** Is the solution that you have found in Question **3.5** global? Can you find other solutions? Justify your answers!

**Q4** Let $n \geq 2$ be a given integer and let $\Omega \subset \mathbb{R}^n$ be a given bounded, connected open set with smooth boundary. Let moreover $f : \mathbb{R} \to \mathbb{R}$ be a given continuous function such that $|f(s)| \leq C_f|s|$ for all $s \in \mathbb{R}$, for some $C_f > 0$. We are interested in the solutions $u : \Omega \to \mathbb{R}$ to the boundary value problem

$$\begin{cases} \Delta^2 u = f(u), & \text{in } \Omega, \\ u = \partial_{x_i} u = 0, \ i \in \{1, \dots, n\}, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

For a four times differentiable function $u : \Omega \to \mathbb{R}$ we use the notation

$$\Delta^2 u := \Delta(\Delta u) = \sum_{i,j=1}^{n} \partial_{x_i}\partial_{x_i}\partial_{x_j}\partial_{x_j} u.$$

**4.1** Show that there exists a constant $C = C(n, \Omega) > 0$ depending only on $\Omega$ and $n$, such that for any $u \in C^2(\overline{\Omega})$ with $u = \partial_{x_i} u = 0,$ on $\partial\Omega,$ $i \in \{1, \dots, n\}$, we have

$$\|\nabla u\|_{L^2(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)}.$$

**4.2** Deduce from Question **4.1** that there exists a constant $\tilde{C} = \tilde{C}(n, \Omega)$ depending only on $\Omega$ and $n$, such that for any $u \in C^2(\overline{\Omega})$ with $u = \partial_{x_i} u = 0,$ on $\partial\Omega,$ $i \in \{1, \dots, n\}$, we have

$$\|u\|_{L^2(\Omega)} \leq \tilde{C}\|\Delta u\|_{L^2(\Omega)}.$$

**4.3** Show that the constant function 0 is always a classical solution to the problem (3).

**4.4** We say that $u \in C^2(\overline{\Omega})$ is a weak solution to (3), if

$$\int_\Omega (\Delta u)(\Delta \varphi) \mathrm{d}\boldsymbol{x} = \int_\Omega f(u)\varphi \mathrm{d}\boldsymbol{x},$$

for all $\varphi \in C^2(\overline{\Omega})$, with $\varphi = \partial_{x_i}\varphi = 0$ on $\partial\Omega,$ $i \in \{1, \dots, n\}.$ Give a sufficient condition involving $n, \Omega$ and $C_f$ that ensures that the problem (3) does not have a nonzero weak solution in the class $C^2(\overline{\Omega})$.

*Hint:* use a specific test function in the weak formulation, then use the previously established inequalities and detect a condition which leads to a contradiction.

**Q5** Let $n \geq 2$ be a given integer, let $\Omega \subset \mathbb{R}^n$ be a given bounded connected open set with smooth boundary $\partial\Omega$. Let moreover $g : \partial\Omega \to \mathbb{R}$ be a given continuous function. We consider the solution to the boundary value problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Recall that $u \in C^1(\overline{\Omega})$ is said to be a weak solution to (4) if $u = g$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi \mathrm{d}\boldsymbol{x} = 0,$$

for all $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$.

**5.1** Show that (4) has at most one weak solution.

**5.2** Determine the energy functional and the minimisation problem for which the weak formulation of (4) can be seen as a first order necessary optimality condition.

**5.3** Fix $\boldsymbol{x}_0 \in \Omega$ and $r, R > 0$ such that $B_r(\boldsymbol{x}_0) \subset B_R(\boldsymbol{x}_0) \subset \Omega$, where $B_\rho(\boldsymbol{y})$ denotes the open ball of radius $\rho > 0$ centered at $\boldsymbol{y}$. Suppose that $u \in C^1(\overline{\Omega})$ is a weak solution to (4). Show that there exists a constant $C > 0$ such that

$$\int_{B_r(x_0)} |\nabla u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \leq \frac{C}{(R-r)^2} \int_{B_R(x_0) \setminus B_r(x_0)} |u(\boldsymbol{x}) - \lambda|^2 \mathrm{d}\boldsymbol{x}, \ \forall \lambda \in \mathbb{R}.$$

*Hint:* consider a special test function of the form $\varphi := (u - \lambda)\eta^2$ and plug it into the weak formulation satisfied by $u$. Here choose $\eta \in C^1_c(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(\boldsymbol{x}_0)$ and $\eta \equiv 0$ on $\Omega \setminus B_R(\boldsymbol{x}_0)$. What property does $\nabla\eta$ need to satisfy? You do not need to construct such an $\eta$ explicitly.