Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH4051-WE01

Title:
General Relativity IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** In a two-dimensional spacetime with coordinates (t, x), metric $ds^2 = -dt^2 + 4x\,dt\,dx$, we have a vector field $V^\mu = (1, x)$.

(i) Compute the components of the corresponding covector field V_μ.

(ii) Define new coordinates $\tilde{t}$, $\tilde{x}$ by $\tilde{t} = t - x^2$, $\tilde{x} = x^2$. Compute the components $\tilde{V}^\mu$ and $\tilde{g}_{\mu\nu}$ of the vector field and metric with respect to the new coordinates.

1.2 Consider the following action in d spacetime dimensions

$$S\left[g_{\mu\nu}, B_{\mu\nu}\right] = -\int d^d x \sqrt{-g} F_{\mu\nu\lambda} F^{\mu\nu\lambda},$$

where $F_{\mu\nu\lambda} = \nabla_{[\mu} B_{\nu\lambda]}$, $B_{\nu\lambda} = -B_{\lambda\nu}$, and the connection is the Levi-Civita connection (metric compatible and torsion free). You may discard any boundary terms that appear.

(i) Obtain the equations of motion for $F_{\mu\nu\lambda}$.

(ii) Obtain the contribution of this matter action to the energy momentum tensor.

Q2 Consider a two-dimensional spacetime with metric

$$ds^2 = \frac{1}{(x^2 + y^2)}(dx^2 + dy^2),$$

and the vector fields

$$V^\mu = (-y, x), \quad W^\mu = (x, y), \quad Z^\mu = (x, -y).$$

2.1 Calculate the Christoffel symbols for this metric.

2.2 Write Killing's equation for this spacetime. For each of these vector fields, determine whether or not it is a Killing vector for this metric.

2.3 The commutator of two vector fields V, W is defined by

$$[V, W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu.$$

Show that the commutator of two vector fields defines a vector field. The commutator thus defines a map which takes two vector fields as inputs and outputs a third. Can this map be described in terms of a type $[1, 2]$ tensor? If so, give an expression for the components of this tensor field. If not, explain why not.

2.4 Can we find new coordinates $\bar{x}, \bar{y}$ such that $V = \partial_{\bar{x}}, Z = \partial_{\bar{y}}$? If so, construct these coordinates explicitly, if not say why not.

Q3 Consider a two-dimensional spacetime with metric

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2,$$

and the curves given by $\sinh\rho = \sin\lambda, \tan t = \alpha\tan\lambda$, where λ is a parameter along the curves, and α labels different curves.

3.1 Calculate the norm of the tangent vector to the curves, as a function of λ and α.

3.2 Find the values of α such that the curves are timelike at the origin $t = \rho = 0$.

3.3 Find the values of α such that the curves are geodesics, with λ an affine parameter.

3.4 Find all the timelike and null geodesics through the origin.

3.5 Find all the points in the spacetime that are within the light cone of the origin; that is, which are connected to the origin by a timelike or null curve.

Q4 Consider the following spherically symmetric metric

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + h(r)r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

where

$$f(r) = \left(1 - \frac{L}{r}\right)^{m+1}, \quad g(r) = \left(1 - \frac{L}{r}\right)^{n-1}, \quad h(r) = \left(1 - \frac{L}{r}\right)^{n},$$

with $L > 0$, m and n real constants. In this problem you will consider the motion of particles in this background. (This metric does not solve Einstein's equations so do not use them.)

4.1 Is this metric asymptotically flat?

4.2 Use conserved quantities to write the geodesic equation for massive particles as

$$\left(\frac{dr}{ds}\right)^2 + V(r) = 0\,,$$

where s is an affine parameter along the geodesic. $V(r)$ should be expressed in terms of the constants of motion and the functions $f(r)$, $g(r)$ and $h(r)$ above. You may take the motion to be on the $\theta = \pi/2$ plane.

4.3 Consider a massive particle released from rest at infinity on the $\theta = \pi/2$ plane. How much proper time elapses along the particle's worldline as the particle moves between some $r = R > L$ and $r = L$? Leave your answer in terms of a definite integral, should one arise.

4.4 This metric is a solution of a modified theory of gravity, where the Einstein-Hilbert action is replaced by the following action

$$S_{\text{gravity}}[g_{\mu\nu}, \phi] = \int d^4x\sqrt{-g}\left(R\phi(x) + F(\phi)\right)\,,$$

where ϕ is a scalar field, and $F(\phi)$ is a function of both ϕ and the metric. In this modified theory the matter action is unchanged, for example a point particle with worldline $X^\mu(s)$ has action

$$S_{p.p.}[g_{\mu\nu}, X^\mu] = \int ds \sqrt{-g_{\mu\nu}\frac{dX^\mu}{ds}\frac{dX^\nu}{ds}} \,.$$

A transformation $\hat{g}_{\mu\nu} = \kappa\phi g_{\mu\nu}$, with κ a constant, transforms $S_{\text{gravity}}[g_{\mu\nu}, \phi]$ to

$$S_{\text{gravity}}[\hat{g}_{\mu\nu}, \phi] = \int d^4x \sqrt{-\hat{g}}\left(\hat{R} + \hat{F}(\phi)\right) \,,$$

where $\hat{R}$ is the Ricci scalar of the metric $\hat{g}$ and $\hat{F}(\phi)$ is $F(\phi)$ after this transformation. This is given, you do not have to derive any of this. The action now looks like the Einstein-Hilbert action for the metric $\hat{g}$ plus a matter action for ϕ. Does this mean this theory is the same as Einstein's gravity? Compare this theory to Einstein's general relativity, namely in terms of the equations for the metric and the equations of motion for matter (including geodesics), before and after the transformation $\hat{g}_{\mu\nu} = \kappa\phi g_{\mu\nu}$. No computations are needed in this problem, just explain in words.

Q5 5.1 Consider the following metric in cylindrical coordinates

$$ds^2 = -dt^2 + dz^2 + \left(1 + \alpha \log\left(\frac{r}{r_0}\right)\right)\left(dr^2 + r^2 d\phi^2\right) \,,$$

where α is a small positive number, $z \in (-\infty, \infty)$, $\phi \in [0, 2\pi)$ and $r \in [0, \infty)$.

(i) Define $u = 1/r$ and obtain an equation for the deflection of light, moving in the $z = 0$ plane, in the form

$$\left(\frac{du}{d\phi}\right)^2 = F(u) \,,$$

where you should work out $F(u)$ and express it in terms of constants of the motion, α and r_0. Do not solve the equation. What happens when $\alpha = 0$?

(ii) Take α to be a small parameter and change the radial coordinate as

$$\left(1 + \alpha \log\left(\frac{r}{r_0}\right)\right) r^2 = (1+\alpha)\bar{r}^2 \,,$$

expanding for small α and keeping the leading terms in α only. Change the angular coordinate to an appropriate $\bar{\phi}$ such that you can write the metric as

$$ds^2 = -dt^2 + dz^2 + d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2 \,.$$

Is this Minkowski space?

5.2 This question is independent from the above. Consider the following spacetime

$$ds^2 = -\left(\alpha^2 r^2 - \frac{b}{r}\right)dt^2 + \frac{1}{\alpha^2 r^2 - \frac{b}{r}}dr^2 + r^2 d\phi^2 + \alpha^2 r^2 dz^2 \,,$$

where b and α are positive real numbers. Two observers are stationary at $z = 0$, $\phi = 0$ and at $r = r_1$ and $r = r_2$ respectively. Assume $r_1 > r_2 > b^{1/3}/\alpha^{2/3}$. The observer at r_2 emits a photon with frequency ω_2 which is observed by the observer at r_1. What frequency ω_1 does the observer at r_1 observe?