Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH4061-WE01

Title:
Advanced Quantum Theory IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 Consider the following theory of two real scalar fields ϕ and σ of mass m and M respectively,

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \frac{1}{2}M^2\sigma^2 - \frac{1}{2}g\sigma\phi^2.$$

1.1 State the Feynman rules needed to compute time-ordered correlation functions in this theory.

In the following two questions, **1.2** and **1.3**, you may assume that the one-point functions of the fields vanish after appropriate regularisation, thus you can ignore any diagrams which contribute to $\langle 0|\hat{\phi}\left(x\right)|0\rangle$ or $\langle 0|\hat{\sigma}\left(x\right)|0\rangle$.

1.2 Consider the three-point function:

$$\langle 0|T\left\{\hat{\phi}\left(x_1\right)\hat{\phi}\left(x_2\right)\hat{\sigma}\left(x_3\right)\right\}|0\rangle \tag{1}$$

(where T denotes time-ordering). Draw the only non-trivial Feynman diagram contributing to (1) which is $O\left(g\right)$ and give the corresponding analytic expression in terms of g and the appropriate propagators.

1.3 Draw all non trivial Feynman diagrams contributing to (1) at $O\left(g^3\right)$. Specify the symmetry factor for each diagram.

1.4 Write down the expression for the generating functional of all correlation functions $Z_g[J_\sigma, J_\phi]$ for this theory as a path integral. Now rewrite $Z_g[J_\sigma, J_\phi]$ in the form of a formal differential operator involving $\delta/\delta J_\phi$ and $\delta/\delta J_\sigma$ acting on the free theory generating functional. Finally write down the result of performing the path integral in the free theory generating functional, you don't need to derive this result and you may ignore an overall normalisation.

1.5 Use these results to compute the generating functional $Z_g[J_\sigma, J_\phi]$ to $O(g)$ and use it to check your answer to part **1.2**. You may express your answers using a graphical notation as long as you explain the meaning of the notation.

1.6 Now set $J_\sigma = 0$ and recompute the path integral $Z_g[0, J_\phi]$ giving your answer in the form of a differential operator involving $\delta/\delta J_\phi$ but not $\delta/\delta J_\sigma$, acting on a functional of the source J_ϕ. Use this to compute the generating functional $Z_g[0, J_\phi]$ to $O(g^2)$ and thus obtain the connected graph contributions to

$$\langle 0|T\left\{\hat{\phi}\left(x_1\right)\hat{\phi}\left(x_2\right)\hat{\phi}\left(x_3\right)\hat{\phi}\left(x_4\right)\right\}|0\rangle$$

and

$$\langle 0|T\left\{\hat{\phi}\left(x_1\right)\hat{\phi}\left(x_2\right)\right\}|0\rangle$$

at this order (again you can express the result graphically).

Q2 Consider a theory of N scalar fields ϕ_a, $a = 1, \ldots N$. Suppose that the Lagrangian density depends on all the ϕ_a and their derivatives $\partial_\mu \phi_a$ and $\partial_\mu \partial_\nu \phi_a$:

$$\mathcal{L} = \mathcal{L}\left(\phi_1, \ldots, \phi_N, \partial_\mu \phi_1, \ldots, \partial_\mu \phi_N, \partial_\mu \partial_\nu \phi_1, \ldots, \partial_\mu \partial_\nu \phi_N\right).$$

2.1 Derive the Euler-Lagrange equations of motion.

2.2 Consider the following infinitesimal transformation:

$$\delta \phi_a = \alpha \Delta \phi_a,$$

where α is a continuous, constant, parameter. Suppose that this is a symmetry of the action S such that $\delta S = 0$. Show that the formula for the corresponding conserved current j^μ is given by

$$j^\mu = \sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \Delta \phi_a + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \partial_\nu \phi_a\right)} \partial_\nu \Delta \phi_a - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \partial_\nu \phi_a\right)}\right) \Delta \phi_a.$$

2.3 Consider the following Lagrangian density for a *complex* scalar field ϕ

$$\mathcal{L} = -\left(\partial_\mu \phi\right)\left(\partial^\mu \phi^*\right) - \left(\partial_\mu \partial_\nu \phi\right)\left(\partial^\mu \partial^\nu \phi^*\right) - m^2 \phi \phi^*.$$

Show explicitly that the following transformation is a symmetry of the action

$$\phi \rightarrow e^{i\alpha} \phi,$$
$$\phi^* \rightarrow e^{-i\alpha} \phi^*,$$

where α is a continuous, constant, parameter. Determine an expression for the current j^μ associated to this symmetry. Show explicitly that the current is conserved.

Q3 A non-interacting real scalar field ϕ of mass m admits the decomposition

$$\phi\left(x\right)=\int\frac{d^{3}\overrightarrow{k}}{\left(2\pi\right)^{3}}\frac{1}{\sqrt{2\omega_{\overrightarrow{k}}}}\left[a_{\overrightarrow{k}}e^{ik\cdot x}+a^{*}_{\overrightarrow{k}}e^{-ik\cdot x}\right]$$

where $\omega_{\overrightarrow{k}}=\sqrt{|\overrightarrow{k}|^{2}+m^{2}}$ and $k=\left(\omega_{\overrightarrow{k}},\overrightarrow{k}\right)$ and $x=(t,\overrightarrow{x})$ are the momentum and position 4-vectors respectively.

Upon quantization, $a_{\overrightarrow{k}}$ and $a^{*}_{\overrightarrow{k}}$ get promoted to operators $\hat{a}_{\overrightarrow{k}}$ and $\hat{a}^{\dagger}_{\overrightarrow{k}}$ which satisfy the commutation relations

$$\left[\hat{a}_{\overrightarrow{k}},\hat{a}^{\dagger}_{\overrightarrow{k}'}\right]=\left(2\pi\right)^{3}\delta^{(3)}\left(\overrightarrow{k}-\overrightarrow{k}'\right),\qquad\left[\hat{a}_{\overrightarrow{k}},\hat{a}_{\overrightarrow{k}'}\right]=0,\qquad\left[\hat{a}^{\dagger}_{\overrightarrow{k}},\hat{a}^{\dagger}_{\overrightarrow{k}'}\right]=0.$$

3.1 The energy E and the momentum $\overrightarrow{P}$ of the field configuration are given by

$$\begin{aligned}E&=\int d^{3}\overrightarrow{x}\,\frac{1}{2}\dot{\phi}^{2}+\frac{1}{2}\left(\overrightarrow{\nabla}\phi\right)\cdot\left(\overrightarrow{\nabla}\phi\right)+\frac{1}{2}m^{2}\phi^{2},\\ \overrightarrow{P}&=-\int d^{3}\overrightarrow{x}\,\dot{\phi}\overrightarrow{\nabla}\phi.\end{aligned}$$

Upon quantization there is an ambiguity in the definition of the corresponding operators $\hat{E}$ and $\overrightarrow{\hat{P}}$. Where does the ambiguity come from? How is the ambiguity fixed? Using your answer, determine $\hat{E}$ and $\overrightarrow{\hat{P}}$ in terms of $\hat{a}_{\overrightarrow{k}}$ and $\hat{a}^{\dagger}_{\overrightarrow{k}}$.

3.2 Explain how particle states are constructed using $\hat{a}_{\overrightarrow{k}}$ and $\hat{a}^{\dagger}_{\overrightarrow{k}}$. In particular, define the vacuum $|0\rangle$, single-particle particle and n-particle states. Your answer should refer to the operators $\hat{E}$ and $\overrightarrow{\hat{P}}$.

Q4 Consider a classical closed string with action

$$S = -\frac{T}{2}\int \mathrm{d}\tau \int \mathrm{d}\sigma\, \sqrt{-h}h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X_\mu\, .$$

Here α and β take on values 0 and 1, and $\mu = 0, \ldots, d-1$.

4.1 Write down the equation of motion for the fields $X^\mu(\tau,\sigma)$ and $h^{\alpha\beta}(\tau,\sigma)$. Show that, for the choice $h^{\alpha\beta} = \mathrm{diag}(-1,1)$, the configuration

$$\begin{aligned} X^0 &= a\,\tau\,, \\ X^1 &= b\,\cos\tau\,\cos\sigma\,, \\ X^2 &= b\,\cos\tau\,\sin\sigma\,, \\ X^3 &= c\,\tau\,. \end{aligned}$$

can be turned into a solution to all equations of motion, provided the constants a, b and c are chosen to satisfy a certain relation. Give this relation.

4.2 The components of the charges of the string with end points σ_L, σ_R are given by

$$P^\mu(\tau) = T\int_{\sigma_L}^{\sigma_R} \mathrm{d}\sigma\, \dot{X}^\mu\,,$$

Compute the value of all components of these charges for the solution (**4.1**) in the two cases

(i) $\sigma_L = 0, \sigma_R = 2\pi$.
(ii) $\sigma_L = 0, \sigma_R = \pi$.

Which of the P^μ are conserved in each case?

4.3 Plot the solution in the X^1, X^2 plane for τ taking fixed values from 0 to π with steps of $\pi/4$, carefully distinguishing the two different cases in **4.2**. What type of boundary condition does the string satisfy in the X_1, X_2 directions in the second case?

4.4 Prove directly from the equations of motion that *any* solution of the equations of motion will satisfy

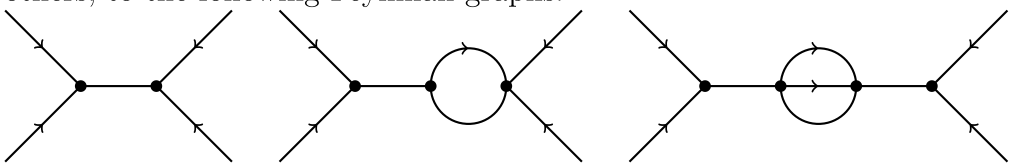
$$P^\mu(\tau_0) - P^\mu(0) = \int_0^{\tau_0} d\tau\, \left(X'^\mu(\tau,\sigma_R) - X'^\mu(\tau,\sigma_L)\right)\ .$$

Now verify by computing both sides of this equation for all values of μ and in both of the cases in **4.2** that this equation is indeed always satisfied by the above solution.

Q5 Consider an interacting real scalar field theory with action

$$S = \int \mathrm{d}^4 x \left[-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \right] .$$

5.1 (i) The computation of a particular scattering amplitude has led, among many others, to the following Feynman graphs:



To which process do these graphs correspond? What is the difference between the three graphs, in as far as the power of $\hbar$ is concerned? Write down the mathematical expression for the *amplitude* corresponding to each graph. Include the symmetry factors. Do *not* evaluate the integrals.

(ii) Explain what is meant by the term "superficial degree of divergence". State the superficial degree of divergence for each divergent graph, and explain your answer. If you cut off all loop integrals at some high momentum scale $|p| = \Lambda$, how would you expect these divergent diagrams to behave for large Λ?

5.2 (i) When d dimensional open string theory is quantised in the lightcone gauge one obtains a Hilbert space which can be built from the vacuum state $|p^i\rangle$ and operators $\hat{\alpha}^j_n$, where $j = 2, .., d-1$ only runs over "transverse modes". Give the commutation relations satisfied by $\hat{\alpha}^j_n$. What is $\hat{\alpha}^j_n |p^i\rangle$ for $n \geq 0$? The mass squared of any state is given by

$$M^2 = \frac{1}{\alpha'}(\sum_{n\geq 1} \hat{\alpha}^i_{-n}\hat{\alpha}^i_n - a)$$

where a is a normal ordering constant (and summation over the index i is assumed). Derive carefully the mass of the states

A. $|p^i\rangle$.

B. $\hat{\alpha}^j_{-1}|p^i\rangle$.

How does your answer help fix the value of a?

(ii) Write down all states that have

A. $M^2 = \frac{1}{\alpha'}$.

B. $M^2 = \frac{2}{\alpha'}$.

C. $M^2 = \frac{3}{\alpha'}$.

How many independent states are there with mass squared $3/\alpha'$? (Give your answer in terms of d but you don't need to simplify the expression).