Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH41220-WE01

Title:
Analysis

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Consider the relation, $\sim$ on $[0,1]$ defined as, $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Prove that $\sim$ is an equivalence relation, namely

(i) Prove that for all $x \in [0,1]$, $x \sim x$ (Reflexive property).

(ii) Prove that for all $x, y \in [0,1]$, if $x \sim y$, then $y \sim x$ (Symmetric property).

(iii) Prove for all $x, y, z \in [0,1]$, if $x \sim y$ and $y \sim z$, then $x \sim z$ (Transitivity property).

1.2 Denote the equivalence class defined by $\sim$ containing x by $[x]_\sim$ and the quotient set by A. Consider the set $E \subset [0,1]$ obtained by choosing a unique point from each equivalence class in A. Let $r \in \mathbb{Q}$, and consider

$$E_r = \{y \in \mathbb{R} \mid y = x + r, x \in E\}.$$

(i) Prove that, for all $r, s \in \mathbb{Q}$, $r \neq s$, $E_s \cap E_r = \emptyset$.

(ii) Denote by $S = \bigcup_{r \in [-1,1] \cap \mathbb{Q}} E_r$ and prove that $[0,1] \subset S \subset [-1,2]$.

1.3 Prove that the set E is not Lebesgue measurable.

Q2 Let $A \subset \mathbb{R}$ be a subset and $x \in \mathbb{R}$ a point. Define $d(x, A) = \inf_{y \in A} |x - y|$. Denote by $\overline{A}$ the closure of A and by A^c the complement of A in $\mathbb{R}$. Show the following.

2.1 For a fixed A the function $f(x) = d(x, A)$ is continuous.

2.2 $\{x \mid d(x, A) = 0\} = \overline{A}$.

2.3 A set A is closed if and only if $d(x, A) > 0$ for any $x \in A^c$.

2.4 Does the statement in 2.2 hold if we define $d(x, A) = \sup_{y \in A} |x - y|$ instead? Justify your response.

Q3 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue integrable, measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ which converge almost everywhere to a limit function

$$f(x) = \lim_{n\to\infty} f_n(x).$$

The limit function f is measurable, integrable and it satisfies the following:

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n.$$

Suppose that for all $n \in \mathbb{N}$, $f_n \geq 0$.

3.1 Consider the auxiliary sequence $g_n := \min(f_n, f)$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n(x)dx = \int_{\mathbb{R}} f(x)dx.$$

3.2 Show that

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - f(x)|dx.$$

Q4 **4.1** Consider the space of functions

$$C^1[-1,1] = \{f : [-1,1] \to \mathbb{R} : f \text{ is differentiable, } f' \text{ is continuous}\},$$

where f' denotes the first derivative of f. Define $\|\cdot\|_1 : C^1[-1,1] \to \mathbb{R}$ as

$$\|f\|_1 = \left(\int_{-1}^{1} (f^2 + (f')^2)\right)^{1/2} = \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2\right)^{1/2}.$$

(i) Prove that $\|\cdot\|_1$ is a norm on $C^1[-1,1]$.

(ii) For $n \in \mathbb{N}$, we define $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}.$$

Does the sequence $(f_n)_n$ converge in $(L^2[-1,1], \|\cdot\|_{L^2})$? Justify your response.

(iii) Does the sequence $(f_n)_n$ converge in $(C^1[-1,1], \|\cdot\|_1)$? Justify your response.

4.2 For $n \in \mathbb{N}$, we define $g_n : \mathbb{R} \to \mathbb{R}$ as

$$g_n(x) = \frac{1}{2n}\sqrt{|\cos(nx)|}\, \chi_{[0,(\pi\sqrt{n})/2]}(x).$$

Is $(g_n)_n$ a Cauchy sequence in $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$? Give a full justification of your response.

4.3 Let $E \subseteq \mathbb{R}$ be measurable. Let $0 < \alpha < 1$ and $1 \leq p \leq q \leq s \leq \infty$ be such that

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{s}.$$

Suppose that $h \in L^p(E) \cap L^s(E)$. Prove that

$$\|h\|_{L^q} \leq \|h\|_{L^p}^{\alpha} \|h\|_{L^s}^{1-\alpha}.$$

Q5 **5.1** Let $k \in \mathbb{N}$ and consider the space of functions

$$C[k, k+1] = \{f : [k, k+1] \to \mathbb{R} : f \text{ is smooth}\}$$

(by smooth we mean infinitely continuously differentiable) with the norm

$$||f||_{\max} = \max_{x \in [k,k+1]} |f(x)|.$$

(i) Does there exist an inner product on $C[0, 1]$ denoted $\langle \cdot, \cdot \rangle$ such that $||f||_{\max} = \sqrt{\langle f, f \rangle}$ for $f \in C[0, 1]$? Justify your response.

(ii) Let $T : C[k, k+1] \to \mathbb{R}$ be the functional defined as

$$T(f) = \frac{d^k}{dx^k} f(x)\bigg|_{x=k+1}.$$

For which $k \in \mathbb{N}$, $k \geq 2$, is T a bounded linear functional on $(C[k, k+1], || \cdot ||_{\max})$? Justify your response.

5.2 For each of the following statements, either provide a proof to show that the statement is true, or construct a counterexample to show that the statement is false.

(i) Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let (x_n) be an orthonormal sequence of $\mathcal{H}$. Then any subsequence of (x_n) converges in $\mathcal{H}$ with respect to the norm $|| \cdot || = \sqrt{\langle \cdot, \cdot \rangle}$.

(ii) Let $\mathcal{H}$ be a separable Hilbert space. Let M be an orthonormal subset of $\mathcal{H}$. Then M is a countable subset of $\mathcal{H}$.

5.3 Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function. Suppose that there exist constants $0 < \beta \leq 1$ and $C > 0$ such that

$$|f(x+h) - f(x)| \leq C|h|^\beta \quad \forall x, h.$$

Recall that the Fourier coefficients of f are defined as

$$a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky}\, dy, \quad k \in \mathbb{Z}.$$

Prove that there exists a constant $A > 0$ such that

$$|a_k(f)| \leq \frac{A}{|k|^\beta}, \quad k \in \mathbb{Z}.$$