EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH4141-WE01

Title:
Geometry IV

Time (for guidance only):	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Write an equation of an ellipse E passing through the point $(1, 0)$ and having foci at $(0, -1)$ and $(0, 1)$. Find the eccentricity and the directrices of E.

1.2 Is there a projective map taking the hyperbola $x^2 - y^2 = 1$ to the ellipse $x^2 + \frac{y^2}{2} = 1$ so that the point $(1, 0)$ is mapped to itself? Justify your answer.

1.3 Is it possible to obtain a parabola isometric to $y = 2x^2$ as a section of the cone $x^2 + y^2 = z^2$? Justify your answer.

1.4 An ellipse E is drawn on the plane, and a point $X \in E$ is marked. Using ruler and compass, construct the tangent line to E at X. Describe and justify the construction.

You can use without proofs and further descriptions the following constructions:
- the midpoint of a given segment;
- the line perpendicular to a given line through a given point.

Q2 **2.1** Let $ABCD$ be a quadrilateral on the unit sphere. Let $\angle ABC = \angle CDA = \pi/2$ and $AB = BC = CD = DA = a$. Find BD.

2.2 Show that there exists a spherical regular pentagon with all angles $2\pi/3$. Find its area.

2.3 Let ABC be a triangle on S^2 with three acute angles (i.e. $\angle A, \angle B, \angle C < \pi/2$). Let $\triangle A'B'C'$ be the polar triangle to $\triangle ABC$. Is it possible that $\triangle A'B'C'$ also has three acute angles? Justify your answer.

2.4 Let f be an isometry of S^2. Suppose that $f \circ r = r \circ f$ for every reflection r. Describe all such isometries f. Justify your answer.

Q3 **3.1** In $\mathbb{R}P^2$, compute the cross-ratio $[A, B, C, D]$ of the points A, B, C, D given by $(0 : 1 : 0)$, $(2 : 1 : 0)$, $(1 : 2 : 0)$, $(1 : 1 : 0)$ respectively.

3.2 Four lines in $\mathbb{R}P^2$ are in general position if no three of them are concurrent. Is it true that the group of projective transformations acts transitively on all quadruples of lines in general position?

3.3 Let a_1, a_2, a_3, a_4 be lines in $\mathbb{R}P^2$ concurrent at the point A, and let b_1, b_2, b_3 be lines concurrent at the point B. Denote $P_{ij} = a_i \cap b_j$ for $i = 1, \dots, 4$, $j = 1, 2, 3$. Suppose that the lines $P_{12}P_{21}, P_{22}P_{31}, P_{32}P_{41}$ are concurrent. Show that the lines $P_{12}P_{23}, P_{22}P_{33}, P_{32}P_{43}$ are concurrent too.

3.4 Formulate the statement dual to the statement in part (3.3) and prove it without referring to the proof of (3.3).

Q4 Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be mutually tangent circles not passing through the same point and let $\mathcal{C}$ be a circle tangent to each of them.

4.1 How many ways are there to chose a circle or line $\mathcal{C}$ so that $\mathcal{C}$ is tangent to each of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$?

4.2 Assume the circle $\mathcal{C}_4$ is disjoint from $\mathcal{C}_2$ and tangent to $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}$. Let $P_i = \mathcal{C}_i \cap \mathcal{C}_{i+1}$ for $i = 1, 2, 3, 4$ (where $\mathcal{C}_5 := \mathcal{C}_1$).
Show that $[P_1, P_2, P_3, P_4]$ is real.

4.3 Under the assumptions of (4.2), let $\mathcal{C}_{013}$ be a circle or line orthogonal to the circles $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_3$, and let $\mathcal{C}_{024}$ be a circle or line orthogonal to $\mathcal{C}, \mathcal{C}_2, \mathcal{C}_4$. Which values can the angle between $\mathcal{C}_{013}$ and $\mathcal{C}_{024}$ take?

4.4 Under the assumptions of (4.2), is there a Möbius transformation which takes circles $\mathcal{C}_2$ and $\mathcal{C}_4$ to concentric circles of radius 1 and 2? Justify your answer.

Q5 Let $ABCD$ be an ideal quadrilateral in $\mathbb{H}^2$.

5.1 Does the group of isometries $Isom(\mathbb{H}^2)$ act transitively on ideal quadrilaterals?

5.2 Let l be a common perpendicular to AB and CD. Show that the lines AC, BD and l are concurrent.

5.3 Suppose that the quadrilateral $ABCD$ has an inscribed circle, i.e. a circle tangent to all four sides. Which values can the radius of the inscribed circle take?

5.4 Assume that $ABCD$ has an inscribed circle. Let r_1, r_2, r_3, r_4 be reflections with respect to AB, BC, CD, DA. Find the type of the isometry $r_4 r_3 r_2 r_1$.

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