

## EXAMINATION PAPER

Examination Session: May/June

2021

Year:

Exam Code:

MATH41420-WE01

Title:

Solitons

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks.				
	Please start each question on a new page. Please write your CIS username at the top of each page.		h page.		
	To receive credit, your answers must show your working and explain your reasoning.				

**Revision:** 

## **Q1 1.1** A field $\phi(x, t)$ has action

$$S[\phi] = \int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx \left( -\phi_x \phi_t - \phi_x^3 + \phi_{xx}^2 \right) .$$

Use the principle of least action to find the Euler-Lagrange equation (or equation of motion) for the field  $\phi$ , assuming that  $\phi(x, t)$  is fixed at  $t = t_1$  and  $t = t_2$ , and that  $\phi_t, \phi_x, \phi_{xx}, \phi_{xxx} \rightarrow 0$  as  $x \rightarrow \pm \infty$ . Show that this equation reduces to the KdV equation  $v_t + 6vv_x + v_{xxx} = 0$  for a second field v(x, t) which is related to  $\phi(x, t)$  in a way that you should find.

**1.2** A field u(x, t) defined on the half-line  $x \le 0$  obeys the equation of motion  $u_{tt} - u_{xx} + \exp(u) = 0$  in the bulk and the boundary conditions

$$x \rightarrow -\infty$$
:  $u_t, u_x, \exp(u) \rightarrow 0$   
 $x = 0$ :  $u_x(0, t) = V'(u(0, t))$ ,

with V a function of the boundary value u(0, t) of the field u(x, t) at x = 0. The densities

$$T_4 = u_{++}^2 + \frac{1}{4}u_{+}^4, \qquad X_2 = -u_{+}^2 \exp(u)$$
  
$$T_{-4} = u_{--}^2 + \frac{1}{4}u_{-}^4, \qquad X_{-2} = -u_{-}^2 \exp(u),$$

which are expressed in terms of derivatives with respect to the light-cone coordinates  $x^{\pm} = \frac{1}{2}(t \pm x)$ , satisfy the equations

$$\partial_- T_4 = \partial_+ X_2$$
,  $\partial_+ T_{-4} = \partial_- X_{-2}$ .

Show that the charge

$$Q = \int_{-\infty}^{0} dx \ (T_4 - X_2 + T_{-4} - X_{-2}) + F(u(0, t), u_t(0, t))$$

is conserved provided that  $V(u(0, t)) = c \exp(u(0, t)/2)$ , where *c* is a constant, and that the boundary contribution  $F(u(0, t), u_t(0, t))$  takes an appropriate form, that you should find.

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**Q2** Consider the following solution of the sine-Gordon equation:

$$u(x, t) = 4 \arctan(t \operatorname{sech}(x))$$
.

- **2.1** Show that this solution contains a kink and an anti-kink.
- **2.2** Find the approximate trajectories  $x_k(t)$  and  $x_{ak}(t)$  of the centres of the kink (k) and the antikink (ak) at early times (large and negative *t*) and late times (large and positive *t*). Sketch these trajectories in the (*x*, *t*) plane. Finally, calculate the velocity  $v_{k/ak}(t) = \dot{x}_{k/ak}(t)$  and the acceleration  $a_{k/ak}(t) = \ddot{x}_{k/ak}(t)$  of the kink and the anti-kink at early and late times, and specify their signs.
- **2.3** Calculate the energy

$$E = \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + 1 - \cos(u) \right]$$

of a single static kink or anti-kink, which is described by the field

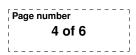
$$u(x, t) = 4 \arctan\left(\exp(\pm(x - x_0))\right) .$$

You may use without proof the integral

$$\int_{-\infty}^{+\infty} dx \, \operatorname{sech}^2(x) = 2 \; .$$

This energy *E* is the mass *M* of the kink or anti-kink.

**2.4** Use the previous results and Newton's law F = Ma (force = mass  $\times$  acceleration) to find how the force F between a kink and an anti-kink depends on the distance d between them, when the distance is large. Is this force attractive or repulsive?



Q3 Let

$$[D_t^m D_x^n(F,G)](x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n F(x,t)G(x',t') \bigg|_{\substack{x' = x \\ t' = t}},$$

where *m*, *n* are non-negative integers and *F*, *G* are any functions of *x* and *t*. It is known that, if a pair of functions f(x, t) and g(x, t) obey the system of equations

$$\begin{cases} (D_x^3 + D_t)(f, g) = 0\\ D_x^2(f, f) + D_x^2(g, g) = 0 \end{cases},$$

then the field u(x, t) given by

$$u = 2 \frac{\partial}{\partial x} \arctan(g/f)$$

is a solution of the mKdV equation  $u_t + 6u^2u_x + u_{xxx} = 0$ .

3.1 Show that

$$u=2\frac{D_x(g,f)}{g^2+f^2}\;.$$

**3.2** Now assume that *f* and *g* take the form

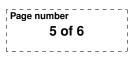
 $f(x,t) = 1 + \epsilon \exp[\theta(x,t)], \qquad g = 1 + \epsilon \exp[\hat{\theta}(x,t)],$ 

where  $\epsilon$  is a formal expansion parameter and

$$\theta(x, t) = ax + bt + c$$
,  $\hat{\theta}(x, t) = \hat{a}x + \hat{b}t + \hat{c}$ ,

with constants  $a, b, c, \hat{a}, \hat{b}, \hat{c}$ . Working order by order in  $\epsilon$ , find a real solution of the system of equations which depends on both x and t.

**3.3** Find the corresponding solution u(x, t) of the mKdV equation.



- **Q4 4.1** Explain, in terms of their actions on other functions, what it means for two differential operators to be equal, and what it means for a differential operator to be multiplicative. If D = d/dx and g(x) is a general function of x, show that  $Dg = gD + g_x$  as differential operators, and derive a formula expressing  $D^2g$  as a sum of terms in which all powers of D appear on the right.
  - **4.2** Let  $L = D^2 + u(x)$ , with u(x) some given function, and let  $B = \alpha(x)D + \beta(x)$ . Giving full details of your calculations, find the most general forms of the functions  $\alpha(x)$  and  $\beta(x)$  such that [L, B] is multiplicative. If *u* also depends on *t* and  $L_t + [L, B] = 0$ , what partial differential equation must *u* satisfy?
  - **4.3** Now let  $M = x^2D^2 + xD + u(x)$ , with u(x) some given function, and  $C = \gamma(x)D$ . Find the most general form of  $\gamma(x)$  such that [M, C] is multiplicative, and write down the partial differential equation for *u* that would follow from setting  $M_t + [M, C]$  equal to zero.
  - **4.4** Show that  $x^2D^2 + xD$  is self-adjoint with respect to the modified inner product

$$\langle f,g\rangle = \int_0^\infty f(x)^*g(x)\frac{dx}{x},$$

when acting on functions on  $(0, \infty)$  for which  $\langle f, f \rangle$  is finite.

**Q5** For this question you can assume that all functions which arise are such that the vanishing of the Wronskian W[f, g] implies that f and g are linearly dependent. Consider the Schrödinger equation

$$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi(x)=k^2\psi(x)$$

where  $V(x) \rightarrow 0$  as  $x \rightarrow \pm \infty$ .

- **5.1** Show that any two bound state eigenfunctions  $\psi_1(x)$  and  $\psi_2(x)$  sharing the same bound state eigenvalue  $k^2 < 0$  must be linearly dependent. (Hint: first show that their Wronskian is constant.)
- **5.2** If V(x) is symmetric, so that V(x) = V(-x), show that all bound state eigenfunctions  $\psi$  are either even ( $\psi(-x) = \psi(x)$ ) or odd ( $\psi(-x) = -\psi(x)$ ). (Hint: first use the result of **5.1** to show that  $\psi(x) \propto \psi(-x)$ , and then consider the possible values of the proportionality constant.)
- **5.3** Now suppose that the potential V(x) from **5.2** is given by

$$V(x) = -a\delta(x+r) - b\delta(x) - a\delta(x-r)$$

where *a*, *b* and *r* are real numbers with r > 0, and  $\delta(x)$  is the Dirac delta function. Setting  $k = i\mu$ ,  $\mu > 0$  and normalising your solutions such that  $\psi(x) \sim e^{-\mu x}$  as  $x \to +\infty$ , write down the general form that even and odd bound state eigenfunctions must take in each of the regions x < -r, -r < x < 0, 0 < x < r and x > r, and the matching conditions that should be imposed at x = -r, x = 0 and x = r. In each case your solutions should depend on just two undetermined parameters *A* and *B*, say, which you can take to be the coefficients of the two exponentials in the region 0 < x < r.

- **5.4** Apply the matching conditions to the odd bound states, and eliminate *A* and *B* to find a constraint on  $\mu$  that should be independent of *b*. Using a graphical method, show that this constraint has no solutions with  $\mu > 0$  for  $a \le 0$ , and one solution with  $\mu > 0$  for a > 0 if (and only if) *a* is larger than a function of *r* that you should find.
- **5.5** Repeat the analysis of **5.4** to find a constraint on the even bound states, and analyse how the number of bound states depends on *a* and *b* when both are positive. You can assume that there are never more than two even bound states.