Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH4151-WE01

Title:
Topics in Algebra and Geometry IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** For $\tau \in \mathbb{H}$ fixed and $z \in \mathbb{C}$ we define

$$h(z) = h_\tau(z) := \prod_{n=1}^{\infty} \left(1 - q^{n-1/2} e^{2\pi i z}\right),$$

where $q = e^{2\pi i \tau}$. Carefully show that $h(z)$ defines an entire function. Determine the zeros of $h(z)$ and their order.

1.2 For a function f on $\mathbb{H}$, and an integer $k \in \mathbb{Z}$, we define the operator

$$(\delta_k f)(\tau) := \frac{\partial f}{\partial \tau}(\tau) + \frac{k}{2i \operatorname{Im}(\tau)} f(\tau), \quad \tau \in \mathbb{H}.$$

Assuming that the derivative $\frac{\partial f}{\partial \tau}$ exists, show that for all $\gamma \in SL_2(\mathbb{Z})$ we have the equality of functions,

$$\delta_k(f|_k\gamma) = (\delta_k f)|_{k+2}\gamma.$$

Here, for an element $\gamma \in SL_2(\mathbb{Z})$, an integer m, and a function $g : \mathbb{H} \to \mathbb{C}$, we define the function $g|_m\gamma : \mathbb{H} \to \mathbb{C}$ by

$$(g|_m\gamma)(\tau) := j(\gamma, \tau)^{-m} g(\gamma\tau).$$

Q2 With the notation as in Q1.1, set $g(z) = h(z)h(-z)$.

2.1 Show

$$g(z+1) = g(z) \qquad \text{and} \qquad g(z+\tau) = -q^{-1/2} e^{-2\pi i z} g(\tau).$$

2.2 Set $f(z) := \frac{g(z+\frac{1}{2})}{g(z)}$. Show $f(z+\tau) = -f(z)$ and conclude that f is an elliptic function for the lattice $2\tau\mathbb{Z} + \mathbb{Z}$.

2.3 Explicitly describe the zeros and poles of f on all of $\mathbb{C}$ and directly verify Theorems Liouville B,C,D for f.

Q3 **3.1** Let $\Omega = \mathbb{Z}\sqrt{-2} + \mathbb{Z}$ and let $\wp(z)$ be the Weierstrass $\wp$-function associated to Ω. Show that $\wp(\sqrt{-2}z)$ is an even elliptic function with respect to Ω. Determine the location of its poles and their order and conclude that $\wp(\sqrt{-2}z)$ has order 4.

3.2 Express $\wp(\sqrt{-2}z)$ as a rational function in $\wp(z)$ using the half-lattice values e_1, e_2, e_3 as coefficients.

Q4 We consider the function

$$\vartheta(\tau) := \theta\left(\frac{\tau}{2}\right) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}, \quad \tau \in \mathbb{H},$$

where $\theta(\tau)$ is the Jacobi theta series. Show that,

4.1 $\vartheta(\tau) + \vartheta(\tau+1) = 2\vartheta(4\tau)$.

4.2 $i\,\vartheta^2\left(1 - \frac{1}{\tau}\right) = \tau(\vartheta(\tau/4) - \vartheta(\tau))^2$.

4.3 Define $f(\tau) := \vartheta^4(\tau) - \vartheta^4(\tau+1) + \tau^{-2}\vartheta^4(1 - \frac{1}{\tau})$. Show that $f(\tau+1) = -f(\tau)$ and $f(-1/\tau) = -\tau^2 f(\tau)$. Conclude that $f(\tau)$ is the zero function.

4.4 Define the function $g(\tau) := \frac{\vartheta^8(\tau)\vartheta^8(\tau+1)\vartheta^8(1-\frac{1}{\tau})}{\tau^4}$. Show that $g(\tau) = \alpha\Delta(\tau)$ for some $0 \neq \alpha \in \mathbb{C}$. Here $\Delta(\tau)$ denotes the discriminant function.

Q5 Let $f \in S_k(\Gamma)$ and $g \in M_\ell(\Gamma)$ be modular forms of weight $k > 0$ and $\ell > 0$ respectively for the modular group $\Gamma = SL_2(\mathbb{Z})$. We write their q-expansions as $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ and $g(\tau) = \sum_{n=0}^{\infty} b_n q^n$, and we assume $a_n, b_n \in \mathbb{R}$ for all n. We set $c_n := a_n b_n$, and define $D(s) := \sum_{n=1}^{\infty} c_n n^{-s}$ with $s \in \mathbb{C}$.

5.1 Show that the series $D(s)$ converges absolutely for $Re(s) > \frac{k}{2} + \ell$. If we further assume that g is also a cusp form, show that the series $D(s)$ converges absolutely for $Re(s) > 1 + \frac{k+\ell}{2}$.

5.2 From now on and for the rest of this question we assume that both f and g are cusp forms and that they are also normalised Hecke eigenforms. For a prime number p, we define $\alpha_p, \beta_p \in \mathbb{C}$ as follows:

$$1 - a_p X + p^{k-1}X^2 = (1-\alpha_p X)(1-\overline{\alpha_p}X), \quad \text{and} \quad 1 - b_p X + p^{\ell-1}X^2 = (1-\beta_p X)(1-\overline{\beta_p}X).$$

Here you can assume that $|a_p| \leq 2p^{\frac{k-1}{2}}$, and $|b_p| \leq 2p^{\frac{\ell-1}{2}}$ for all p. Show that for all primes p, and $n \in \mathbb{N}$ we have,

$$a_{p^n} = \sum_{m=0}^{n} \alpha_p^m \overline{\alpha_p}^{n-m}.$$

5.3 With notation as above, show the formal power series identity:

$$\sum_{n=0}^{\infty} c_{p^n} X^n = \frac{1 - p^{k+\ell-2}X^2}{(1-\alpha_p\beta_p X)(1-\alpha_p\overline{\beta_p}X)(1-\overline{\alpha_p}\beta_p X)(1-\overline{\alpha_p}\overline{\beta_p}X)}.$$

5.4 Show that $D(2 + \frac{k+\ell}{2}) \neq 0$.