Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH4161-WE01

Title:
Algebraic Topology IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Let $X = S^1 \times D^2$ be the solid torus and let $\partial X = S^1 \times S^1$ be the torus boundary. Compute the homology groups (with integer coefficients) of the quotient space $X/\partial X$.

1.2 Let W be a compact manifold of dimension $2n+1$ with boundary $\partial W = M$. Show that for the Euler characteristic we have $\chi(M) = 2\chi(W)$.

Show that the projective plane $\mathbb{R}\mathbf{P}^2$ is not the boundary of a compact manifold of dimension 3.

1.3 Let M be a closed orientable manifold of dimension $4n+2$. Show that the rank of $H_{2n+1}(M)$ (with integer coefficients) cannot be 1. You may use standard results on cup-products without proof.

Q2 **2.1** For $p \in \mathbb{R}^3$ and $r > 0$, let

$$S^2_{p,r} = \{x \in \mathbb{R}^3 : |x - p| = r\}$$

be the 2-sphere in $\mathbb{R}^3$ with centre p and radius r.

Compute the homology (with integer coefficients) of the space $X \subset \mathbb{R}^3$ defined by

$$X = S^2_{(0,0,0),2} \cup S^2_{(1,0,0),1} \cup S^2_{(-1,0,0),1}.$$

2.2 We define the space $X \subset \mathbb{R}^5$ as the following union of four subsets of $\mathbb{R}^5$

$$\begin{aligned} X = &\{p \in \mathbb{R}^5 : |p| = 1\} \\ &\cup\{(r,0,0,0,0) \in \mathbb{R}^5 : -1 \leq r \leq 1\} \\ &\cup\{(0,s,t,0,0) \in \mathbb{R}^5 : s^2 + t^2 \leq 1\} \\ &\cup\{(0,0,t,u,v) \in \mathbb{R}^5 : t^2 + u^2 + v^2 \leq 1\}. \end{aligned}$$

Compute the homology (with integer coefficients) of the space X.

Q3 **3.1** Consider the set $X = \{1,2,3\}$ which we topologize by giving it the topology

$$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}.$$

Show that X is contractible (in other words homotopy equivalent to the space $\{\mathrm{p}\}$ consisting of a single point) via the homotopy equivalence

$$\{\mathrm{p}\} \to X : \mathrm{p} \mapsto 3.$$

3.2 Hence or otherwise, compute the homology groups (with integer coefficients) of the space $Y = \{1,2,3,4\}$ which has the topology

$$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}.$$

Q4 Let (M, d) be a metric space, and for $k \geq 0$ an integer let $\Gamma^k(M, d)$ be the set of all functions $\varphi\colon M^{k+1} \to \mathbb{Z}$, where $M^{k+1} = M \times M \times \cdots \times M$ is the cartesian product of $k + 1$ factors of M. Define $\delta^k : \Gamma^k(M, d) \to \Gamma^{k+1}(M, d)$ by

$$\delta^k(\varphi)(x_0, \dots, x_{k+1}) = \sum_{i=0}^{k+1}(-1)^i\varphi(x_0, \dots, \hat{x}_i, \dots, x_{k+1}),$$

where $\hat{x}_i$ means that x_i is to be omitted from the formula.

4.1 Show that $(\Gamma^*(M, d), \delta^*)$ is a cochain complex.

4.2 For $\varepsilon > 0$ let $\Gamma^k_\varepsilon(M, d) \subset \Gamma^k(M, d)$ consist of those functions for which $\varphi(x_0, \dots, x_k) = 0$ whenever $d(x_i, x_j) < \varepsilon$ for all $i, j \in \{0, \dots, k\}$ (for $k = 0$ this means $\varphi(x) = 0$ for all $x \in M$). Show that $\Gamma^*_\varepsilon(M, d)$ with the restriction of δ^* is also a cochain complex.

4.3 Let $\overline{\Gamma}^k_\varepsilon(M, d) = \Gamma^k(M, d)/\Gamma^k_\varepsilon(M, d)$ and $\bar{\delta}^k : \overline{\Gamma}^k_\varepsilon(M, d) \to \overline{\Gamma}^{k+1}_\varepsilon(M, d)$ the coboundary induced by δ^k. Write

$$\mathcal{H}^k_\varepsilon(M, d) = \ker \bar{\delta}^k / \operatorname{im} \bar{\delta}^{k-1},$$

and calculate $\mathcal{H}^k_\varepsilon(P, d)$ for all integers $k \geq 0$, where P consists of one point, and d is the unique metric on P.

4.4 Calculate $\mathcal{H}^0_\varepsilon(T, d)$ for all $\varepsilon > 0$, where $T = \{x_0, x_1\}$ is a set of two elements, and d is an arbitrary metric on T.

4.5 Assume that (M, d) is connected. Show that $\mathcal{H}^0_\varepsilon(M, d) \cong \mathbb{Z}$ for all $\varepsilon > 0$.

4.6 Give an example of a metric space (M, d) such that for all $\varepsilon > 0$ the group $\mathcal{H}^0_\varepsilon(M, d)$ is different from $H^0(M; \mathbb{Z})$, the 0-th singular cohomology group of M.

Q5 Let G, H, K be abelian groups.

We say that G satisfies property (S), if whenever $\varphi\colon G \to H$ is a homomorphism, and $\psi\colon K \to H$ is a surjective homomorphism, then there exists a homomorphism $\chi\colon G \to K$ with $\varphi = \psi \circ \chi$.

We say that G satisfies property (R), if whenever $\varphi\colon H \to G$ is a homomorphism, and $\psi\colon H \to K$ is an injective homomorphism, then there exists a homomorphism $\chi\colon K \to G$ with $\varphi = \chi \circ \psi$.

5.1 Show that G satisfies property (S) if and only if $\mathrm{Ext}(G, A) = 0$ for all abelian groups A. Hint: For one direction, express property (S) in terms of $\psi_*\colon \mathrm{Hom}(G, K) \to \mathrm{Hom}(G, H)$.

5.2 Show that G satisfies property (R) if and only if $\mathrm{Ext}(A, G) = 0$ for all abelian groups A.

5.3 Assume that G satisfies property (R), and let H be a subgroup of G. Show that G/H has property (R).

5.4 Give an example each of a group G such that

- G has property (S), but not property (R).
- G does not have property (S) nor property (R).
- G has property (S) and property (R).

Justify your statements, and state every result you use from the course.