Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH4171-WE01

Title:
Riemannian Geometry IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

ED01/2021

Q1 **1.1** Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = x^2 + y^2 - z^2$. Determine for which values of $c \in \mathbb{R}$ the level sets $f^{-1}(c) = \{\,(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = c\,\}$ are smooth submanifolds of $\mathbb{R}^3$.

1.2 Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map with $n > k$. Let $y \in f(\mathbb{R}^n)$ be a regular value of f, let $M = f^{-1}(y)$, and fix $p \in M$. Show that $T_pM = \ker(df_p)$.

1.3 The *Heisenberg group* is given by

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Note that H has a global coordinate chart $\varphi\colon H \to \mathbb{R}^3$ given by

$$\varphi(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}) = (x, y, z).$$

Let $e = \varphi^{-1}(0,0,0)$, $g = \varphi^{-1}(x,y,z)$, and $L_g, R_g\colon H \to H$ be the left- and right- multiplication maps, defined by $L_g(h) = gh$ and $R_g(h) = hg$ for $h \in H$. Calculate the tangent vectors

$$dL_g(e)\left(\frac{\partial}{\partial y}|_e\right) \in T_gH$$

and

$$dR_g(e)\left(\frac{\partial}{\partial z}|_e\right) \in T_gH$$

in terms of $\frac{\partial}{\partial x}|_g$, $\frac{\partial}{\partial y}|_g$, and $\frac{\partial}{\partial z}|_g$.

Q2 A set $A \subset \mathbb{R}^2$ has *measure zero* if, for every $\varepsilon > 0$, there is a sequence $B_1, B_2, \ldots$ of closed rectangles in $\mathbb{R}^2$ with

$$A \subset \cup_{n=1}^{\infty} B_n$$

and

$$\Sigma_{n=1}^{\infty} \mathrm{vol}(B_n) < \varepsilon,$$

where $\mathrm{vol}(B_n)$ is the usual area of B_n as a subset of $\mathbb{R}^2$. A subset A of a smooth 2-dimensional manifold M has *measure zero* if there is a sequence of charts $\{(U_i, \varphi_i)\}_{i=1}^{\infty}$, with $A \subset \cup_{i=1}^{\infty} U_i$ such that each set $\varphi_i(A \cap U_i) \subset \mathbb{R}^2$ has measure zero. You may assume without proof the following:

(M0) If $f\colon M \to N$ is a smooth function between smooth manifolds M, N and $V \subset N$ is open, then the preimage $f^{-1}(V) \subset M$ is open.

(M1) If $f\colon U \subset \mathbb{R}^2 \to W \subset \mathbb{R}^2$ is a smooth function between open subsets U and W of $\mathbb{R}^2$, and $A \subset U$ has measure zero, then $f(A) \subset W$ has measure zero.

(M2) Subsets of sets of measure zero in $\mathbb{R}^2$ have measure zero.

(M3) The countable union of sets of measure zero in $\mathbb{R}^2$ is a set of measure zero.

2.1 Show that if $A \subset M$ has measure zero, then $\varphi(A \cap U) \subset \mathbb{R}^2$ has measure zero for any chart (U, φ) of M.

2.2 Assume, without proof that, if M is connected, then the converse to item 2.1 holds, i.e., if $A \subset M$ is such that $\varphi(A \cap U) \subset \mathbb{R}^2$ has measure zero for any chart (U, φ) of M, then A has measure zero. Show that if $f\colon M \to N$ is a diffeomorphism between two 2-dimensional connected smooth manifolds and $A \subset M$ has measure zero, then $f(A) \subset N$ has measure zero.

Q3 Let $M = \{(x, y, z) \mid x^2+y^2 = z\} \subset \mathbb{R}^3$ be the paraboloid equipped with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^3$ and consider the coordinate chart

$$\varphi = (r, \alpha)\colon M \setminus \{(x, 0, x^2) \mid x \geq 0\} \to (0, \infty) \times (0, 2\pi)$$

given by $\varphi^{-1}(r, \alpha) = (r \cos \alpha, r \sin \alpha, r^2)$.

3.1 Calculate in these coordinates

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \alpha}, \quad \nabla_{\frac{\partial}{\partial \alpha}} \frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial \alpha}} \frac{\partial}{\partial \alpha},$$

where ∇ is the Levi-Civita connection.

3.2 Compute the vector fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial \alpha}$ in terms of $x, y, z, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

3.3 Compute the scalar curvature of M with its induced metric at $p = (0, 0, 0) \in M$ using the coordinates x, y of M.

Q4 **4.1** Let $a, b, c, d, B \in \mathbb{R}$ with $ad - bc = 1$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then define the map $\phi_{A,B} : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ for $(z, r) \in \mathbb{C} \times \mathbb{R}$ by

$$\phi_{A,B}(z, r) = (\frac{az + b}{cz + d},\ r + B).$$

Now consider the collection $G := \{\phi_{A,B} | \ A, B \text{ as as above}\}$. Prove that G is a Lie group with composition of mappings as group operation.

4.2 Let G be as in (4.1) and $e \in G$ be the neutral element. Let $v := (D, 1)$ with $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Prove that $v \in T_eG$ and then calculate explicitly the left-invariant vector field X on G such that $X(e) = v$.

4.3 Let $M = \{(x_1 + ix_2, x_3) \in \mathbb{C} \times \mathbb{R} | \ x_2 > 0\}$ and $g_{11} = g_{22} = x_2^{-2}$, $g_{12} = g_{21} = g_{13} = g_{31} = g_{23} = g_{32} = 0$, $g_{33} = 1$ define the Riemannian manifold (M, g). Prove that the elements of G are isometries of (M, g).

4.4 Let (M, g) be as in (4.3). Prove that the parametrized curve

$$x(t) = (x_1(t), x_2(t), x_3(t)) \in M$$

for $t \in [0, a]$ is a geodesic of (M, g) if and only if the following two properties hold:
(i) the parametrized curve $(x_1(t), x_2(t)) \in (\mathbb{R}^2 \cap \{x_2 > 0\}, g_{ij}, i, j = 1, 2)$ is a

geodesic;
(ii) the parametrized curve $x_3(t) \in (\mathbb{R}, g_{33})$ is a geodesic.

Q5 **5.1** Let $v, x \in \mathbb{R}^n$ and let $x \cdot v$ be their Euclidean inner product. Fix coordinates x_i, $i = 1, ..., n$, of $\mathbb{R}^n$ and set $g_{kk} := e^{x \cdot v}$ for $k = 1, ..., n$, $g_{ij} = 0$ for $i \neq j$. This defines a Riemannian metric g on $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$ and $P := \mathrm{span}\{\partial/\partial \mathrm{x}_1, \partial/\partial \mathrm{x}_2\} \subset \mathrm{T_p}\mathbb{R}^\mathrm{n}$. Compute the sectional curvature of P in $(\mathbb{R}^n, g)$.

5.2 Let v, g be chosen as in (5.1) and let $w \in \mathbb{R}^n$, $|w| = |v|$, and let $h_{kk} := e^{x \cdot w}$ for $k = 1, ..., n$, $h_{ij} = 0$ for $i \neq j$ define a Riemannian metric h on $\mathbb{R}^n$. Prove that there exists an isometry $\phi : (\mathbb{R}^n, g) \to (\mathbb{R}^n, h)$.

5.3 Let g_{ij} be the Euclidean metric on $\mathbb{R}^n$, let $f : \mathbb{R}^n \to \mathbb{R}^{>0}$ be a positive differentiable function, and let $h_{ij} := f g_{ij}$ be another metric on $\mathbb{R}^n$. Assume that every geodesic curve in $(\mathbb{R}^n, g)$ is also a geodesic curve in $(\mathbb{R}^n, h)$. Prove or disprove the following claim : The function f is a constant function.