

EXAMINATION PAPER

Examination Session: May/June

Year: 2021

Exam Code:

MATH4171-WE01

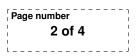
Title:

Riemannian Geometry IV

| Time (for guidance only): | 3 hours | |
|-------------------------------|---------|---|
| Additional Material provided: | | |
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| Materials Permitted: | | |
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| Calculators Permitted: | Yes | Models Permitted: There is no restriction on the model of calculator which may be used. |

| Instructions to Candidates: | Credit will be given for your answers to all questions. All questions carry the same marks. | | | | |
|-----------------------------|--|---------------------------------|---|--|--|
| | Please start each question on a new page. Please write your CIS username at the top of each page. | | | | |
| | To receive credit, your answers must explain your reasoning. | wers must show your working and | | | |
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Revision:





- **Q1** 1.1 Let $f: \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + y^2 z^2$. Determine for which values of $c \in \mathbb{R}$ the level sets $f^{-1}(c) = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = c\}$ are smooth submanifolds of \mathbb{R}^3 .
 - **1.2** Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map with n > k. Let $y \in f(\mathbb{R}^n)$ be a regular value of f, let $M = f^{-1}(y)$, and fix $p \in M$. Show that $T_pM = \text{ker}(df_p)$.
 - **1.3** The *Heisenberg group* is given by

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Note that *H* has a global coordinate chart $\varphi \colon H \to \mathbb{R}^3$ given by

$$\varphi\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = (x, y, z).$$

Let $e = \varphi^{-1}(0, 0, 0)$, $g = \varphi^{-1}(x, y, z)$, and $L_g, R_g: H \to H$ be the left- and right- multiplication maps, defined by $L_g(h) = gh$ and $R_g(h) = hg$ for $h \in H$. Calculate the tangent vectors

$$dL_g(e)\left(rac{\partial}{\partial y}|_e
ight)\in T_gH$$

and

$$d {\it R}_{g}({\it e}) \left(rac{\partial}{\partial z} |_{{\it e}}
ight) \in {\it T}_{g} {\it H}$$

in terms of $\frac{\partial}{\partial x}|_g$, $\frac{\partial}{\partial y}|_g$, and $\frac{\partial}{\partial z}|_g$.

Q2 A set $A \subset \mathbb{R}^2$ has *measure zero* if, for every $\varepsilon > 0$, there is a sequence $B_1, B_2, ...$ of closed rectangles in \mathbb{R}^2 with

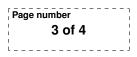
$$A \subset \cup_{n=1}^{\infty} B_n$$

and

$$\sum_{n=1}^{\infty} \operatorname{vol}(B_n) < \varepsilon,$$

where vol(B_n) is the usual area of B_n as a subset of \mathbb{R}^2 . A subset A of a smooth 2-dimensional manifold M has *measure zero* if there is a sequence of charts $\{(U_i, \varphi_i)\}_{i=1}^{\infty}$, with $A \subset \bigcup_{i=1}^{\infty} U_i$ such that each set $\varphi_i(A \cap U_i) \subset \mathbb{R}^2$ has measure zero. You may assume without proof the following:

- (M0) If $f: M \to N$ is a smooth function between smooth manifolds M, N and $V \subset N$ is open, then the preimage $f^{-1}(V) \subset M$ is open.
- (M1) If $f: U \subset \mathbb{R}^2 \to W \subset \mathbb{R}^2$ is a smooth function between open subsets U and W of \mathbb{R}^2 , and $A \subset U$ has measure zero, then $f(A) \subset W$ has measure zero.
- (M2) Subsets of sets of measure zero in \mathbb{R}^2 have measure zero.
- (M3) The countable union of sets of measure zero in \mathbb{R}^2 is a set of measure zero.



- **2.1** Show that if $A \subset M$ has measure zero, then $\varphi(A \cap U) \subset \mathbb{R}^2$ has measure zero for any chart (U, φ) of M.
- **2.2** Assume, without proof that, if *M* is connected, then the converse to item 2.1 holds, i.e., if $A \subset M$ is such that $\varphi(A \cap U) \subset \mathbb{R}^2$ has measure zero for any chart (U, φ) of *M*, then *A* has measure zero. Show that if $f: M \to N$ is a diffeomorphism between two 2-dimensional connected smooth manifolds and $A \subset M$ has measure zero, then $f(A) \subset N$ has measure zero.
- **Q3** Let $M = \{(x, y, z) \mid x^2 + y^2 = z\} \subset \mathbb{R}^3$ be the paraboloid equipped with the Riemannian metric induced by the Euclidean metric on \mathbb{R}^3 and consider the coordinate chart

$$arphi = (r, lpha) \colon M \setminus \{(x, 0, x^2) \mid x \geq 0\}
ightarrow (0, \infty) imes (0, 2\pi)$$

given by $\varphi^{-1}(r, \alpha) = (r \cos \alpha, r \sin \alpha, r^2)$.

3.1 Calculate in these coordinates

$$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial \alpha}, \quad \nabla_{\frac{\partial}{\partial \alpha}}\frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial \alpha}}\frac{\partial}{\partial \alpha},$$

where ∇ is the Levi-Civita connection.

- **3.2** Compute the vector fields $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \alpha}$ in terms of $x, y, z, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.
- **3.3** Compute the scalar curvature of *M* with its induced metric at $p = (0, 0, 0) \in M$ using the coordinates *x*, *y* of *M*.
- **Q4** 4.1 Let $a, b, c, d, B \in \mathbb{R}$ with ad bc = 1, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then define the map $\phi_{A,B} : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ for $(z, r) \in \mathbb{C} \times \mathbb{R}$ by

$$\phi_{A,B}(z,r)=(\frac{az+b}{cz+d},\ r+B).$$

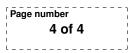
Now consider the collection $G := \{\phi_{A,B} | A, B \text{ as as above}\}$. Prove that *G* is a Lie group with composition of mappings as group operation.

- **4.2** Let *G* be as in (4.1) and $e \in G$ be the neutral element. Let v := (D, 1) with $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Prove that $v \in T_e G$ and then calculate explicitly the left-invariant vector field *X* on *G* such that X(e) = v.
- **4.3** Let $M = \{(x_1 + ix_2, x_3) \in \mathbb{C} \times \mathbb{R} | x_2 > 0\}$ and $g_{11} = g_{22} = x_2^{-2}, g_{12} = g_{21} = g_{13} = g_{31} = g_{23} = g_{32} = 0, g_{33} = 1$ define the Riemannian manifold (M, g). Prove that the elements of *G* are isometries of (M, g).
- **4.4** Let (M, g) be as in (4.3). Prove that the parametrized curve

$$x(t) = (x_1(t), x_2(t), x_3(t)) \in M$$

for $t \in [0, a]$ is a geodesic of (M, g) if and only if the following two properties hold:

(i) the parametrized curve $(x_1(t), x_2(t)) \in (\mathbb{R}^2 \cap \{x_2 > 0\}, g_{ij}, i, j = 1, 2)$ is a





geodesic;

(ii) the parametrized curve $x_3(t) \in (\mathbb{R}, g_{33})$ is a geodesic.

- **Q5 5.1** Let $v, x \in \mathbb{R}^n$ and let $x \cdot v$ be their Euclidean inner product. Fix coordinates x_i , i = 1, ..., n, of \mathbb{R}^n and set $g_{kk} := e^{x \cdot v}$ for k = 1, ..., n, $g_{ij} = 0$ for $i \neq j$. This defines a Riemannian metric g on \mathbb{R}^n . Let $p \in \mathbb{R}^n$ and $P := \operatorname{span}\{\partial/\partial x_1, \partial/\partial x_2\} \subset T_p\mathbb{R}^n$. Compute the sectional curvature of P in (\mathbb{R}^n, g) .
 - **5.2** Let v, g be chosen as in (5.1) and let $w \in \mathbb{R}^n$, |w| = |v|, and let $h_{kk} := e^{x \cdot w}$ for k = 1, ..., n, $h_{ij} = 0$ for $i \neq j$ define a Riemannian metric h on \mathbb{R}^n . Prove that there exists an isometry $\phi : (\mathbb{R}^n, g) \to (\mathbb{R}^n, h)$.
 - **5.3** Let g_{ij} be the Euclidean metric on \mathbb{R}^n , let $f : \mathbb{R}^n \to \mathbb{R}^{>0}$ be a positive differentiable function, and let $h_{ij} := fg_{ij}$ be another metric on \mathbb{R}^n . Assume that every geodesic curve in (\mathbb{R}^n, g) is also a geodesic curve in (\mathbb{R}^n, h) . Prove or disprove the following claim : The function *f* is a constant function.