Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH41820-WE01

Title:
Fluid Mechanics

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

Q1 **1.1** Starting from Faraday's law $\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E}$, Ampère's law $\mu_0 \mathbf{J} = \boldsymbol{\nabla} \times \boldsymbol{B}$, and Ohm's law $\mathbf{J} = \sigma(\mathbf{E} + \boldsymbol{u} \times \boldsymbol{B})$, with σ constant, derive the MHD induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \Delta \boldsymbol{B}$$

and write down η in terms of the other constants.

1.2 If $\boldsymbol{A}$ is a vector potential satisfying $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, show that

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \times \boldsymbol{B} - \frac{1}{\sigma}\mathbf{J} + \boldsymbol{\nabla}\phi$$

for an arbitrary scalar function $\phi(\boldsymbol{x}, t)$.

1.3 Now consider the magnetic helicity in a fixed spherical volume V, defined by

$$K = \int_V \boldsymbol{A} \cdot \boldsymbol{B} dV.$$

Show that

$$\frac{dK}{dt} = 2\int_V \boldsymbol{B} \cdot \frac{\partial \boldsymbol{A}}{\partial t} dV - \oint_{\partial V} \boldsymbol{A} \times \frac{\partial \boldsymbol{A}}{\partial t} \cdot d\boldsymbol{S}$$

1.4 If on the boundary ∂V, $\boldsymbol{u} = \boldsymbol{B}$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, show further that

$$\frac{dK}{dt} = -\frac{2}{\sigma}\int_V \mathbf{J} \cdot \boldsymbol{B} dV + \frac{1}{\sigma}\oint_{\partial V} \boldsymbol{A} \times \mathbf{J} \cdot d\boldsymbol{S}.$$

Q2 Consider a two dimensional incompressible, inviscid flow $\boldsymbol{u}(x, y, t)$ with zero body force. The vorticity of this flow at $t = 0$ is given by

$$\boldsymbol{\omega} = 2\pi k \delta(x - x_0)\delta(y - y_0)\boldsymbol{e}_z,$$

where $x_0, y_0 > 0$.

2.1 Write down (without proof) a steady flow $\boldsymbol{u}(x, y)$ with this vorticity in an infinite domain where $|\boldsymbol{u}| \to 0$ at infinity.

2.2 Two walls are introduced at $x = 0$ and $y = 0$. Find the flow in the semi-infinite domain $(0 < x < \infty, 0 < y < \infty)$, and confirm your answer satisfies appropriate boundary conditions.

2.3 Assume now that initially $y_0 \gg x_0$. By considering the fluid motion at (x_0, y_0), describe how the line vortex moves relative to the walls.

2.4 Does the strength of the line vortex change in time as it moves? Explain your answer.

2.5 Assume instead that $x_0 = y_0 = a$ and both walls are still present. Could a flow be constructed such that the line vortex remains stationary? If so, find such a flow solution.

Q3 Consider a flow which is parallel and uniformly proportional to its vorticity, i.e. a flow with $\boldsymbol{\omega} = \lambda \boldsymbol{u}$ where λ is a constant.

3.1 Show that such a flow is a solution of the steady state, unforced Euler equations, and find the associated pressure (stating any assumptions you are making).

3.2 Assume now that the flow is cylindrically symmetric and of the form

$$\boldsymbol{u}(r,\theta,z) = \frac{1}{r}\boldsymbol{\nabla}\psi \times \boldsymbol{e}_\theta + u_\theta(r,z)\boldsymbol{e}_\theta$$

where $\psi = \psi(r,z)$. Show that $\boldsymbol{\omega} \times \boldsymbol{u} = 0$ implies that

$$ru_\theta = G(\psi)$$

where G is some arbitrary function and that ψ is a solution of

$$-\boldsymbol{\nabla} \cdot \left(\frac{1}{r^2}\boldsymbol{\nabla}\psi\right) = \frac{1}{r^2}G\frac{dG}{d\psi}$$

3.3 Taking ψ of the form $\psi(r,z) = rf(r)$ along with the assertion that $G = -k\psi$, find a well behaved flow solution that satisfies $\boldsymbol{\omega} \times \boldsymbol{u} = 0$.

Q4 Consider a closed glass tank containing two ideal, incompressible, irrotational fluids that meet at an interface. The tank is in a lift accelerating (possibly rapidly) towards the ground. We model the tank as two dimensional so that in the frame of reference of the tank the domain is given by $D = \{(x,z) : 0 < x < L, -h < z < h\}$, the interface is given by $z = \eta(x,t)$, and the body force $\mathbf{f} = A\mathbf{e}_z - g\mathbf{e}_z$, where g and A are the acceleration due to gravity and the movement of the frame, respectively. The density of the fluid above the interface is ρ_1 and below is ρ_2 and you may assume that A is constant.

4.1 The linearised equations of motion satisfied by the velocity potentials are

$$\begin{aligned}
\Delta\phi_1 &= 0, \quad z > 0,\\
\Delta\phi_2 &= 0, \quad z < 0,\\
\frac{\partial\eta}{\partial t} = \frac{\partial\phi_1}{\partial z} &= \frac{\partial\phi_2}{\partial z} \quad \text{at z = 0,}
\end{aligned}$$

along with one further condition at the interface. Find and linearise this condition.

4.2 Trying solutions of the form $\phi(x,z,t) = p(z)q(x)e^{-i\omega t}$ find the dispersion relation for linear waves on the interface.

4.3 In general, when does a dispersion relation indicate that the system is linearly unstable?

4.4 Under what circumstances is this system linearly unstable? Explain how the fluids evolve and find the associated growth rate.

Q5 Consider a flow falling under gravity between two concentric vertical rotating cylinders. The inner cylinder has a radius of a and rotates at angular velocity Ω_1. The outer cylinder has a radius of b and rotates at angular velocity Ω_2, where $b > a$. Assume that $\boldsymbol{u} = F(r)\boldsymbol{e}_z + G(r)\boldsymbol{e}_\theta$ and that $p = p(r)$.

5.1 Show that the incompressible Navier-Stokes equations reduce to

$$\frac{dp}{dr} = \rho_0 \frac{G^2}{r}, \quad \frac{d}{dr}\left(r\frac{dG}{dr}\right) = \frac{G}{r}, \quad \frac{d}{dr}\left(r\frac{dF}{dr}\right) = \frac{\rho_0 g r}{\mu}.$$

5.2 Solve to find $F(r)$ and $G(r)$ subject to appropriate boundary conditions.

5.3 Assume now that $a\Omega_1 = b\Omega_2$. Find the radius at which the slowest fluid rotation occurs.

5.4 Still assuming that $a\Omega_1 = b\Omega_2$, explain in terms of the action of viscosity why the flow rotates slower yet falls faster away from the walls of the cylinders.