Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2021	MATH4241-WE01

Title: Representation Theory IV

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. To receive credit, your answers must show your working and explain your reasoning.

Revision:	

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In this exam, all representations are finite dimensional over the complex numbers.

Q1 Decide, with proof, whether each of the following statements is true or false.

1.1 If G is a nonabelian finite group and p is a prime dividing $|G|$, then G has an irreducible character of degree p.

1.2 If G is a nonabelian subgroup of $\mathrm{GL}_2(\mathbb{C})$, then any matrix $A \in M_2(\mathbb{C})$ such that

$$Ag = gA$$

for all $g \in G$ is a scalar multiple of the identity.

1.3 Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a closed subgroup with Lie algebra $\mathfrak{g}$. If $\exp(iX) \in G$ for all $X \in \mathfrak{g}$, then G is a complex Lie group.

1.4 If V is a finite dimensional $\mathbb{C}$-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$ such that every weight of V occurs with multiplicity one, then V is irreducible.

Q2 **2.1** Show that if χ is a character of a finite group G and $g \in G$ has order two, then

$$\chi(g) \in \mathbb{Z}.$$

A group G of size 72 has conjugacy classes and two characters χ_1 and χ_2 as shown. Each conjugacy class is labeled with the order of its elements (so elements of C_3 have order 3, elements of C_{4A} have order 4, etc.).

class	C_1	C_2	C_3	C_{4A}	C_{4B}	C_{4C}
size	1	9	8	18	18	18
χ_1	1	1	1	-1	1	-1
χ_2	1	1	1	1	-1	-1

2.2 Show that the character table of G is as follows:

class	C_1	C_2	C_3	C_{4A}	C_{4B}	C_{4C}
size	1	9	8	18	18	18
χ_0	1	1	1	1	1	1
χ_1	1	1	1	-1	1	-1
χ_2	1	1	1	1	-1	-1
χ_3	1	1	1	-1	-1	1
χ_4	2	-2	2	0	0	0
χ_5	8	0	-1	0	0	0

2.3 Decompose the character $\Lambda^2\chi_5$ into irreducibles.

2.4 Show that G has a normal subgroup H of index 8, and that G/H is not isomorphic to D_4.

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Q3 Let $(\rho, \mathbb{C}^2)$ be the unique irreducible two-dimensional representation of the dihedral group D_3, and let ϵ be the nontrivial character of D_3 of degree one.

Let $K = D_3 \times D_3$ and let $(\sigma, \mathbb{C}^2)$ be the representation of K defined by

$$\sigma((g_1, g_2)) = \rho(g_1)\epsilon(g_2)$$

for all $g_1, g_2 \in D_3$.

3.1 Show that σ is irreducible.

3.2 Write down the row of the character table of K giving the character of σ.

Now let H be the group of order 72 generated by K and by an element c such that $c^2 = e$ and $c(g_1, g_2) = (g_2, g_1)c$ for all $(g_1, g_2) \in K$. Thus K is a normal subgroup of H of index two and every element of H may be written uniquely as either

$$(g_1, g_2)$$

or

$$(g_1, g_2)c$$

for $g_1, g_2 \in D_3$.

Let

$$\pi = \mathrm{Ind}_K^H(\sigma)$$

and let χ be the character of π.

3.3 Show that π is irreducible.

3.4 If $r \in D_3$ is rotation by $2\pi/3$ and $s \in D_3$ is a reflection, find

$$\chi((r, s))$$

and

$$\chi(c).$$

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Q4 Let $\mathcal{P}^\ell$ be the space of homogeneous complex polynomials of degree ℓ in x, y, and z, regarded as a representation of SO(3) by the formula

$$(gf)(\mathbf{x}) = f(g^T\mathbf{x})$$

for

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and for all $g \in \mathrm{SO}(3)$, $f \in \mathcal{P}^\ell$.

Let $\mathcal{H}^\ell \subset \mathcal{P}^\ell$ be the subspace of harmonic polynomials, which is an irreducible representation of SO(3) (and therefore also of $\mathfrak{so}_3$ and $\mathfrak{so}_{3,\mathbb{C}}$).

Let

$$J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}_3.$$

4.1 Decompose $\Lambda^2\mathcal{H}^3$ as a direct sum of irreducible representations $\mathcal{H}^\ell$ of SO(3).

4.2 State a formula for the action of J_z on $\mathcal{P}^\ell$.

4.3 Find a nonzero element $L \in \mathfrak{so}_{3,\mathbb{C}}$ such that

$$[J_z, L] = -iL,$$

and prove that L takes weight vectors for J_z of weight ki to weight vectors for J_z of weight $(k-1)i$.

4.4 Write down a highest weight vector for J_z in $\mathcal{H}^\ell$.

4.5 When $\ell = 3$, find nonzero weight vectors for J_z in $\mathcal{H}^\ell$ of weights $2i$ and i.

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Q5 Let $\mathfrak{g} = \mathfrak{sl}_{3,\mathbb{C}}$ and for $1 \leq i, j \leq 3$ distinct let $E_{ij} \in \mathfrak{g}$ be the matrix with a '1' in row i and column j, and zeros elsewhere. Let $V = \mathbb{C}^3$ be the standard representation of $\mathfrak{sl}_{3,\mathbb{C}}$.

5.1 Draw weight diagrams for the representations $V \otimes V$ and $V \otimes V^*$, showing your working.

5.2 If v is a weight vector in a representation (ρ, W) of $\mathfrak{g}$ of weight α, let $L(v)$ be the subspace spanned by

$$\rho(E_{32})\rho(E_{21})v,$$
$$\rho(E_{21})\rho(E_{32})v,$$

and

$$\rho(E_{31})v.$$

Show that $L(v) \subset W_\beta$ for some weight β, where W_β is the weight space of weight β.

5.3 Show that $\dim L(v) \leq 2$. Find an example of a representation W and nonzero weight vectors v_0, v_1, v_2 in W such that

$$\dim L(v_0) = 0,$$
$$\dim L(v_1) = 1,$$

and

$$\dim L(v_2) = 2.$$

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