Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH1031-WE01

Title:
Discrete Mathematics

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

Q1 (a) (i) How many arrangements are there of the letters COMBINATION?

(ii) How many arrangements have the two Os adjacent?

(iii) How many arrangements have no two adjacent letters the same?

(b) (i) State the pigeon-hole principle.

(ii) There are n people attending a party ($n \geq 2$). During the party, some pairs of guests shake hands, but no pair of people shake hands more than once. Show that there are two guests who shook the same number of hands during the party.

Q2 (a) Let c_n denote the number of integer solutions $(x_1, x_2, x_3, x_4, x_5, x_6)$ to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = n,$$

with each $x_i \geq 1$, $n \geq 0$ integer.

(i) Derive a formula for c_n via a combinatorial argument.

(ii) Write down a generating function for c_n, and express it as compactly as possible.

(iii) Evaluate $\sum_{n=6}^{\infty} \binom{n-1}{5} 2^{-n}$.

(b) (i) Suppose that the sequence a_n satisfies the recurrence relation

$$a_n = -a_{n-1} + 6a_{n-2}, \quad (n \geq 2),$$

with initial conditions $a_0 = 0$, $a_1 = 3$. Find the generating function $g(x) = \sum_{n=0}^{\infty} a_n x^n$ for the sequence a_n. Hence, or otherwise, find a_n as a function of n.

(ii) Find a_n satisfying the recurrence relation

$$a_n = -a_{n-1} + 6a_{n-2} - 4n + 3, \quad (n \geq 2),$$

with initial conditions $a_0 = 0$, $a_1 = 3$. (For this part of the question, you are not required to find the generating function.)

Q3 (a) A florist stocks roses, lilies, peonies, tulips, delphiniums, chrysanthemums, and azaleas. Bouquets are made up of between 3 and 8 flowers of each type and contain 30 flowers in total. In how many different ways can bouquets be constituted?

(b) Consider the identity

$$\prod_{k=1}^{n} \binom{3k-1}{2} = \frac{(3n)!}{n!\,6^n}, \quad (n \geq 1).$$

(i) Prove the identity by induction.

(ii) Give a combinatorial explanation of the identity. *Hint:* consider arranging people into triples.

Q4 (a) (i) Define the *degree* of a vertex of a graph.

(ii) Prove that in a simple graph $G = (V, E)$ where $|V| > 1$, there must exist vertices $u \neq v$ for which $d(u) = d(v)$.

(b) (i) Given two simple graphs G, H, define what is meant for G and H to be *isomorphic*.

(ii) Define the complement $\overline{G}$.

(iii) Show that $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

(c) (i) Suppose that G and H are simple graphs such that $G \cong H$, where G contains a vertex v with $d(v) = k$ for some $k \in \mathbb{N}$. Show that H contains a vertex w with $d(w) = k$.

(ii) Consider a simple graph $G = (V, E)$ with n vertices. Given a vertex v of G, let $d(v)$ denote the degree of v in G and $\overline{d(v)}$ denote the degree of v in $\overline{G}$. Write down an expression for $\overline{d(v)}$ in terms of $d(v)$ and n.

(iii) Let $\overline{E}$ denote the edge set of $\overline{G}$. Write down an expression for the number of edges of $\overline{G}$, that is, $|\overline{E}|$, in terms of $|E|$ and n.