EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH1061-WE01

Title:
Calculus I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

Q1 **1.1** Without using Taylor series or L'Hôpital's rule, calculate

$$\lim_{x\to 9} \frac{x^2-8x-9}{3-\sqrt{x}}.$$

1.2 Use Taylor series to calculate

$$\lim_{x\to 0} \frac{e^{2x^2}-\cos x}{\sin(3x^2)}.$$

Q2 Calculate

$$\int_{-\pi}^{\pi} \frac{x^3 \sin x}{1+e^{x\cos x}}\, dx.$$

Q3 Let D be the triangle in the (x,y)-plane with vertices $(0,0),(1,3),(3,1)$. Calculate

$$\iint_D 2e^{-y-x}\, dxdy.$$

Q4 Find the functions $y(x)$ and $z(x)$ that satisfy the initial conditions $y(0)=z(0)=0$ and the differential equations

$$y'+2y+z=e^{-x} \quad \text{and} \quad z'-3y-2z=0.$$

Q5 The function $f(x)$ has period 4, that is $f(x+4)=f(x)$, and is given by

$$f(x)=\begin{cases} -1 & \text{if } -2<x<0 \\ 2 & \text{if } \quad 0\le x<2. \end{cases}$$

Calculate the Fourier series of $f(x)$ and use the result to calculate

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$

Q6 6.1 Find the equation of the tangent plane to the surface

$$f(x,y,z) = \left(x^2 + y^2 + z^2\right) e^{-x-2y-z}$$

at the point $(x,y,z) = (1,-1,1)$.

6.2 Find and classify the stationary points of the function

$$f(x,y) = (x^2 + y^2)e^{-x-2y}.$$

Q7 7.1 A change of coördinates is defined as follows:

$$x = 2u + 3v$$
$$y = u + v.$$

Use the Chain Rule to express the derivatives f_x and f_y in terms of f_u and f_v. At the origin $(u,v) = (0,0)$, the function f is such that $(f_u, f_v) = (1,-1)$. Find the slope of the function f in the direction of the unit vector $\hat{n} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ at the corresponding point $(x,y) = (0,0)$ when f is considered as a function of x and y.

7.2 Find the Taylor expansion of the function

$$f(x,y) = x\sin(xy) + \ln\left(x + y - \frac{\pi}{2}\right)$$

about the point $(x,y) = (1, \pi/2)$ to quadratic order.

Q8 A differential operator $\mathcal{L}$ is defined to be

$$\mathcal{L} = (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx}.$$

(a) Show that

$$\mathcal{L}y = \sqrt{1-x^2}\frac{d}{dx}\left(\sqrt{1-x^2}\frac{dy}{dx}\right).$$

(b) Show that the operator $\mathcal{L}$ is self-adjoint with respect to the inner product

$$(f,g) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)g(x)dx$$

where f, g, f_x, g_x are finite at $x = \pm 1$.

(c) By substitution, find a solution to the equation

$$\mathcal{L}y = \lambda y$$

in the form $y = P_2(x) = 1 + Bx^2$ where B and λ are non-zero constants whose values you should determine.

(d) By making the substitution $x = \sin(\theta)$ or otherwise, show that the integral

$$\int_{-1}^{1} \frac{P_2(x)}{\sqrt{1-x^2}} dx = 0.$$

Why would you have expected this to be the case?

Q9 A function $u(x,y)$ is defined in the region $0 \le x \le \pi$, $0 \le y \le \pi$. The function satisfies Laplace's equation $u_{xx} + u_{yy} = 0$, and the boundary conditions $u(0,y) = u(\pi,y) = 0$ together with the boundary conditions $u_y(x,0) = 0$ and $u(x,\pi) = 1$.

(a) Use the method of separation of variables to show that the only non-vanishing solutions of the form $u(x,y) = X(x)Y(y)$ to Laplace's equation which satisfy the three boundary conditions $u(0,y) = u(\pi,y) = u_y(x,0) = 0$ are proportional to

$$u_n(x,y) = \sin(nx)\cosh(ny)$$

where n is a positive integer.

(b) Use the principle of superposition to find $u(x,y)$ which also satisfies the fourth boundary condition $u(x,\pi) = 1$.

Q10 A function $u(x,t)$ obeys the differential equation

$$u_t = u_{xx} + 2u_x + u$$

and initially $u(x,0) = R(x)$.

(a) By taking the Fourier transform of the above equation show that the Fourier transform

$$\tilde{u}(p,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ipx}dx,$$

can be written in the form

$$\tilde{u}(p,t) = \tilde{R}(p)\tilde{G}(p,t)$$

where $\tilde{R}(p)$ is the Fourier transform of $R(x)$, and $\tilde{G}(p,t)$ is a function you should determine.

(b) Find the solution $u(x,t)$ in the form of the convolution of $R(x)$ and another function $G(x,t)$ which you should determine explicitly . You may use that the Fourier transform of $F(x) = e^{-bx^2-cx}$ is

$$\tilde{F}(p) = \sqrt{\frac{\pi}{b}}e^{-(p-ic)^2/4b}.$$