Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH1071-WE01

Title:
Linear Algebra I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Use a separate answer book for each Section. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

Q1 **1.1** Showing your working, find all unit vectors in $\mathbb{R}^4$ which are orthogonal (using the standard dot product) to all three of the vectors

$$\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\\0\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}.$$

1.2 Suppose Π_1 and Π_2 are planes in $\mathbb{R}^3$ with normals $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$. If Π_1 passes through the origin, and Π_2 contains the point $\begin{pmatrix}2\\0\\0\end{pmatrix}$, find the line of their intersection. Show your working.

Q2 **2.1** Showing your working, use row operations to find the inverse of the matrix

$$\begin{pmatrix}1 & 2 & 3\\2 & 5 & 3\\1 & 0 & 8\end{pmatrix}.$$

2.2 Determine the values of a for which the following system of equations have (i) no solutions (ii) exactly one solution, and (iii) infinitely many solutions. Briefly justify your answers.

$$\begin{aligned} x + 2y + z &= 2\\ 2x - 2y + 3z &= 1\\ x + 2y - az &= a\,. \end{aligned}$$

Q3 **3.1** Consider the following three elements of $\mathbb{R}[x]_3$

$$p(x) = x^2 - 2x + 4, \qquad q(x) = 4x^2 + 5, \qquad r(x) = 2x^2 + 4x - 3\,.$$

Show that they are not linearly independent. Find a linearly independent subset of them and find a basis of $\mathbb{R}[x]_3$ which contains your subset. Justify that your set is a basis, quoting any results from the module that you need.

3.2 If the real vector space $V \neq \{\mathbf{0}\}$ is spanned by a finite set of vectors, prove that it has a basis. You may **not** quote any results from lectures, but must prove the result directly from the definitions.

Q4 **4.1** Let V be a real vector space. Define what it means to say that the subset $U \subset V$ is a vector subspace of V.

4.2 Now consider V to be the vector space $M_2(\mathbb{R})$ of 2 by 2 matrices with real entries. Let $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and consider the vector subspaces

$$U = \left\{ A \in M_2(\mathbb{R}) \mid A = A^t \right\} \quad \text{and} \quad W = \left\{ A \in M_2(\mathbb{R}) \mid EAE = A^t \right\} .$$

Write down a basis for each of U, W, $U+W$ and $U \cap W$, and hence or otherwise find the dimensions of these subspaces. You may use any results you need from lectures but must state clearly what you use.

Q5 **5.1** Let a be a real number. Define the function $e_a \colon \mathbb{R}[x]_3 \to \mathbb{R}$ by $e_a(p) = p(a)$ (that is, the value of e_a on the element $p(x) \in \mathbb{R}[x]_3$ is the value of that polynomial at a).

(i) Prove that e_a is a linear map.

(ii) What is its rank and what is its nullity?

(iii) Give a basis for the kernel of e_a.

You may use any results from lectures provided you state them clearly.

5.2 Suppose $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map given by a matrix A. Write T^3 for T composed with itself three times: $T^3 = T \circ T \circ T \colon \mathbb{R}^n \to \mathbb{R}^n$. If $\ker(T^3)$ contains more than just the zero vector, prove that $\det A = 0$. You may use any results from lectures provided you state them clearly.

Q6 Given the matrix

$$A = \begin{pmatrix} 8 & -14 & -7 \\ -7 & 15 & 7 \\ 14 & -28 & -13 \end{pmatrix} ,$$

find a matrix M such that $M^{-1}AM$ is diagonal.

Q7 Consider the linear transformation $S : \mathbb{R}[x]_3 \to \mathbb{R}[x]_3$ defined by

$$\mathcal{S}(p(x)) = \frac{x^2(p(x+1) - 2p(x) + p(x-1))}{2} - p'(x) ,$$

with $p(x) \in \mathbb{R}[x]_3$ and $p'(x) = dp(x)/dx$. Write down the matrix M representing $\mathcal{S}$ on $\mathbb{R}[x]_3$, using the standard basis $\{x^0, x^1, x^2, x^3\}$, and then find the eigenvalues, the eigenvectors, and the eigenfunctions.

Q8 **8.1** Determine, justifying your answer, whether or not the expression

$$(\underline{x}, \underline{y}) = x_1y_1 + x_1y_2 + x_1y_3$$

defines a real inner product on $V = \mathbb{R}^3$;

8.2 Determine, justifying your answer, for which values of $\lambda \in \mathbb{C}$ the expression

$$\langle \underline{z}, \underline{w} \rangle = z_1\bar{w}_1 + \frac{(1+i\sqrt{3})}{2} z_1\bar{w}_2 + \frac{(1-i\sqrt{3})}{2} z_2\bar{w}_1 + \lambda z_2\bar{w}_2$$

defines an hermitian inner product on $V = \mathbb{C}^2$;

8.3 Determine, justifying your answer, whether the expression

$$(A, B) = \mathrm{tr}(B^tA)$$

defines a real inner product on $V = M_n(\mathbb{R})$, where B^t denotes the transpose of B and tr denotes the trace, i.e. $\mathrm{tr}(A) = \sum_{i=1}^n A_{ii}$.

Q9 **9.1** Given two matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = 0$, show that if $\underline{v}$ is an eigenvector of A with eigenvalue λ, then $B\underline{v}$ is also an eigenvector of A with the same eigenvalue λ.

9.2 Show that every matrix $C \in M_n(\mathbb{R})$ can be written as the sum of two matrices C_+ and C_-, one of which is symmetric: $C_+ = (C_+)^t$ and the other is skew-symmetric $C_- = -(C_-)^t$.

Q10 Show that the set

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a, b, c, d \in \mathbb{Z}\,,\ ad - bc = 1\,, c \equiv 0\,(\mathrm{mod}\,2) \right\},$$

forms a group with respect to matrix multiplication. You may assume that matrix multiplication is associative.