

## **EXAMINATION PAPER**

Examination Session: May/June

2022

Year:

Exam Code:

MATH1071-WE01

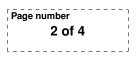
Title:

Linear Algebra I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Use a separate answer book for each Section.	
	Students must use the mathematics specific answer book.	

**Revision:** 





**Q1 1.1** Showing your working, find all unit vectors in  $\mathbb{R}^4$  which are orthogonal (using the standard dot product) to all three of the vectors

$$\begin{pmatrix} 1\\0\\1\\1 \end{pmatrix}, \quad \begin{pmatrix} 2\\1\\0\\1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1\\-1\\1\\-1 \end{pmatrix}$$

- **1.2** Suppose  $\Pi_1$  and  $\Pi_2$  are planes in  $\mathbb{R}^3$  with normals  $\begin{pmatrix} 1\\ 1\\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 1\\ -1\\ 1 \end{pmatrix}$ . If  $\Pi_1$  passes through the origin, and  $\Pi_2$  contains the point  $\begin{pmatrix} 2\\ 0\\ 0 \end{pmatrix}$ , find the line of their intersection. Show your working.
- Q2 2.1 Showing your working, use row operations to find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

2.2 Determine the values of a for which the following system of equations have(i) no solutions (ii) exactly one solution, and (iii) infinitely many solutions. Briefly justify your answers.

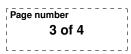
$$\begin{array}{rcl}
x + 2y + z &=& 2\\
2x - 2y + 3z &=& 1\\
x + 2y - az &=& a.
\end{array}$$

**Q3 3.1** Consider the following three elements of  $\mathbb{R}[x]_3$ 

$$p(x) = x^2 - 2x + 4,$$
  $q(x) = 4x^2 + 5,$   $r(x) = 2x^2 + 4x - 3.$ 

Show that they are not linearly independent. Find a linearly independent subset of them and find a basis of  $\mathbb{R}[x]_3$  which contains your subset. Justify that your set is a basis, quoting any results from the module that you need.

**3.2** If the real vector space  $V \neq \{0\}$  is spanned by a finite set of vectors, prove that it has a basis. You may **not** quote any results from lectures, but must prove the result directly from the definitions.





- **Q4** 4.1 Let V be a real vector space. Define what it means to say that the subset  $U \subset V$  is a vector subspace of V.
  - **4.2** Now consider V to be the vector space  $M_2(\mathbb{R})$  of 2 by 2 matrices with real entries. Let  $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and consider the vector subspaces

$$U = \left\{ A \in M_2(\mathbb{R}) \mid A = A^t \right\} \quad \text{and} \quad W = \left\{ A \in M_2(\mathbb{R}) \mid EAE = A^t \right\}.$$

Write down a basis for each of U, W, U+W and  $U \cap W$ , and hence or otherwise find the dimensions of these subspaces. You may use any results you need from lectures but must state clearly what you use.

- **Q5 5.1** Let *a* be a real number. Define the function  $e_a : \mathbb{R}[x]_3 \to \mathbb{R}$  by  $e_a(p) = p(a)$  (that is, the value of  $e_a$  on the element  $p(x) \in \mathbb{R}[x]_3$  is the value of that polynomial at *a*).
  - (i) Prove that  $e_a$  is a linear map.
  - (ii) What is its rank and what is its nullity?
  - (iii) Give a basis for the kernel of  $e_a$ .

You may use any results from lectures provided you state them clearly.

- **5.2** Suppose  $T: \mathbb{R}^n \to \mathbb{R}^n$  is a linear map given by a matrix A. Write  $T^3$  for T composed with itself three times:  $T^3 = T \circ T \circ T : \mathbb{R}^n \to \mathbb{R}^n$ . If ker $(T^3)$  contains more than just the zero vector, prove that det A = 0. You may use any results from lectures provided you state them clearly.
- Q6 Given the matrix

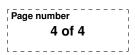
$$A = \begin{pmatrix} 8 & -14 & -7 \\ -7 & 15 & 7 \\ 14 & -28 & -13 \end{pmatrix},$$

find a matrix M such that  $M^{-1}AM$  is diagonal.

**Q7** Consider the linear transformation  $S : \mathbb{R}[x]_3 \to \mathbb{R}[x]_3$  defined by

$$\mathcal{S}(p(x)) = \frac{x^2(p(x+1) - 2p(x) + p(x-1))}{2} - p'(x),$$

with  $p(x) \in \mathbb{R}[x]_3$  and p'(x) = dp(x)/dx. Write down the matrix M representing S on  $\mathbb{R}[x]_3$ , using the standard basis  $\{x^0, x^1, x^2, x^3\}$ , and then find the eigenvalues, the eigenvectors, and the eigenfunctions.





Q8 8.1 Determine, justifying your answer, whether or not the expression

$$(\underline{x}, y) = x_1 y_1 + x_1 y_2 + x_1 y_3$$

defines a real inner product on  $V = \mathbb{R}^3$ ;

**8.2** Determine, justifying your answer, for which values of  $\lambda \in \mathbb{C}$  the expression

$$\langle \underline{z}, \underline{w} \rangle = z_1 \overline{w}_1 + \frac{(1+i\sqrt{3})}{2} z_1 \overline{w}_2 + \frac{(1-i\sqrt{3})}{2} z_2 \overline{w}_1 + \lambda z_2 \overline{w}_2$$

defines an hermitian inner product on  $V = \mathbb{C}^2$ ;

8.3 Determine, justifying your answer, whether the expression

$$(A,B) = \operatorname{tr}(B^t A)$$

defines a real inner product on  $V = M_n(\mathbb{R})$ , where  $B^t$  denotes the transpose of B and tr denotes the trace, i.e.  $\operatorname{tr}(A) = \sum_{i=1}^n A_{ii}$ .

- **Q9** 9.1 Given two matrices  $A, B \in M_n(\mathbb{R})$  such that AB BA = 0, show that if  $\underline{v}$  is an eigenvector of A with eigenvalue  $\lambda$ , then  $\underline{Bv}$  is also an eigenvector of A with the same eigenvalue  $\lambda$ .
  - **9.2** Show that every matrix  $C \in M_n(\mathbb{R})$  can be written as the sum of two matrices  $C_+$  and  $C_-$ , one of which is symmetric:  $C_+ = (C_+)^t$  and the other is skew-symmetric  $C_- = -(C_-)^t$ .
- **Q10** Show that the set

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 \pmod{2} \right\},\$$

forms a group with respect to matrix multiplication. You may assume that matrix multiplication is associative.