Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH1561-WE01

Title:
Single Mathematics A

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

Q1 (i) Use the derivative of $\cos x$ to find $\dfrac{d}{dx}(\arccos x)$.

(ii) Express the complex number $\dfrac{3+5i}{1-3i}$ in the form $a+ib$ with a and b real.

(iii) Use the definitions of sinh and cosh in terms of exp to show that

$$\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B\ .$$

(iv) Find $\dfrac{d}{dx}(x^{\sin x^2})$.

Q2 (i) Evaluate the indefinite integral

$$\int \frac{x+3}{x^2+4x+6}\,dx\ .$$

Hint: $\displaystyle\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan(x/a)$.

(ii) Evaluate the indefinite integral

$$\int \frac{x-1}{(x+1)(x-2)}\,dx\ .$$

(iii) Evaluate the definite integral

$$\int_0^{\pi/4} x^2 \sin x\,dx\ .$$

Q3 (i) Find all complex solutions of the equation

$$z^4 = \sqrt{3} - i$$

and draw them on the complex plane.

(ii) Evaluate the following limits without using l'Hopital's rule. You will only get full marks if you explain carefully the steps and rules you are using to prove the limits.

$$\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9}\ , \qquad \lim_{x\to\infty} \frac{2x+1}{3x+\sin x}\ , \qquad \lim_{x\to 0} \frac{\tan 3x}{x}\ .$$

Q4 (a) Define convergence for an infinite series $S = \sum_{n=1}^{\infty} a_n$. Give a necessary condition for convergence.

(b) Determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{1+\cos n}{\sqrt{n^3+n}}, \qquad \sum_{n=1}^{\infty} (n+1)^2 e^{-n}$$

converge.

(c) Determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty}(n^2-1)(3x)^n.$$

Determine whether the series converges at the endpoints of the interval of convergence.

Q5 (a) For the function $f(x) = e^{\cos x}$,

(i) Find the second-order Taylor polynomial $p_2(x)$ about $x = \frac{\pi}{2}$.

(ii) Use the Lagrange form of the remainder to obtain a bound on the error $f(x) - p_2(x)$ for $x \in (0, \frac{\pi}{2})$.

(b) (i) Find the value of c for which the following three vectors are not linearly independent

$$\begin{pmatrix} c \\ 7 \\ 1 \\ -4 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 1 \\ -2 \end{pmatrix}$$

(ii) Assume c does not have the value you found above. By considering the standard basis vectors of $\mathbb{R}^4$, or otherwise, write down a vector v such that the set of v and three vectors in the previous part of the question form a basis of $\mathbb{R}^4$. How do you know the set is a basis?

Q6 For which values of $\alpha \in \mathbb{R}$ does the system of linear equations

$$\begin{aligned} \alpha x + \alpha y + \alpha^2 z &= \alpha \\ x + \alpha^2 y + \alpha z &= 2 \\ x + y + 2\alpha z &= 1 \end{aligned}$$

have (a) no solutions, (b) infinitely many solutions, (c) a unique solution?

Find all the solutions in the cases (b) and (c) and in each of these two cases write if the set of solutions represents a point, a line, or a plane.

Q7 (a) Find the values of a, b, $c \in \mathbb{C}$ for which the following matrix has real eigenvalues and orthogonal eigenvectors.

$$\begin{pmatrix} 1 & 0 & b \\ c & 2 & a \\ 1-i & 0 & 1 \end{pmatrix}$$

Name the property of the matrix which ensures that it has real eigenvalues and orthogonal eigenvectors

(b) Find the eigenvalues and eigenvectors for the matrix with the values of a, b and c that you found above, and show explicitly that the eigenvectors are orthogonal.