Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH1571-WE01

Title:
Single Mathematics B

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

Q1 (a) The points A, B, C and D have Cartesian coordinates $(1, 5, 4)$, $(3, 7, 1)$, $(3, 2, 3)$ and $(2, 5, 5)$ respectively.

(i) Find the parametric equations of two lines, one passing through A and B, and the other passing through C and D.

(ii) Find the shortest distance between these lines.

(iii) Find the volume of a parallelepiped with edges AB, AC and AD.

(b) In $\mathbb{R}^3$ the spherical polar basis vectors are

$$\begin{aligned} \mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \\ \mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}\,. \end{aligned}$$

(i) What does it mean for a set of vectors to be *orthonormal*?

(ii) Show that the set of spherical polar basis vectors is orthonormal.

(iii) In spherical polar coordinates (r, θ, ϕ) the angles θ and ϕ are functions of time. Find expressions for $\dot{\mathbf{e}}_r$, $\dot{\mathbf{e}}_\theta$ and $\dot{\mathbf{e}}_\phi$ in terms of spherical polar coordinates and spherical polar basis vectors.

Q2 (a) The population $P(t)$ of bacteria obeys the differential equation

$$\frac{dP}{dt} = -2P + 100t.$$

If at $t = 0$ the population is $P(0) = 200$, calculate the population $P(t)$ at later times, and use your result to find an expression for the minimum population at later times.

(b) The functions $u(x)$ and $v(x)$ satisfy the coupled first order differential equations

$$\begin{aligned} u' &= u + 4v + 4e^{-x} \\ v' &= 2u - v + e^{x}. \end{aligned}$$

Find $u(x)$ and $v(x)$ if $u(0) = 4$ and $v(0) = -1$.

Q3 A forced damped simple harmonic oscillator obeys the equation

$$\ddot{x} + 2\dot{x} + (n^2 + 1)x = \sin(nt)$$

where n is a positive constant. If $x(0) = \dot{x}(0) = 0$ find $x(t)$. Show that in the limit of large t, $x(t)$ settles down to an oscillation of amplitude X, where

$$X = \frac{1}{\sqrt{4n^2 + 1}}.$$

You may assume that $\alpha\cos(\kappa t) + \beta\sin(\kappa t)$ is an oscillation of amplitude $\sqrt{\alpha^2 + \beta^2}$.

Q4 (a) A function $f(x)$ is periodic with period 2π and is defined in the interval $-\pi \leq x \leq \pi$ to be $f(x) = x^2$. Find the Fourier series for $f(x)$, and use your result and Parseval's Theorem to evaluate the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(b) Find the general solution to the PDE

$$u_{xx} + y^2 u = 0,$$

and find the particular solution that satisfies the boundary conditions

$$u(0, y) = y^2$$
$$u_x(0, y) = y^2.$$

Q5 An examiner is trying to write a question all about surfaces and critical points, based on a hillside that has a functional form

$$f(x, y) = y^4 + x^4 - 2x^2 - 2y^2 - ax^2y^2$$

where a is some non-negative constant. The idea is that the function should have stable minima, and the students would be asked to find them and all the other critical points. However the examiner finds that not all values of a work. If a is too big for example there are no minima. What values of a would give good minima? Find all the critical points in terms of a and determine what kind they are. *[Hint: you should find that there can be 9 critical points in total.]*

Q6 (a) A spherical ball of radius 1, centred at the origin, is filled with fluid that has a density that varies with height z as $\rho = 1 + bz + az^2$, where a and b are positive constants. Determine

(i) the total mass.

(ii) the centre of mass.

(b) A complex function is given by $f(x, y) = U + iV$ where $U = x^3 - 3y^2x$ and $V = 3yx^2 - y^3$. Determine if the function is analytic or not.

Q7 Peter's Perfect Pie Shop sells pies to students. Customers arrive throughout the day at random intervals.

(a) First, we are interested in the random variable T, which represents the time in minutes between successive customers arriving at the shop. Peter lets us know that the distribution of T has the density function

$$f(t) = \begin{cases} \alpha \exp(-\frac{1}{2}t) & t > 0 \\ 0 & \text{otherwise} \end{cases} .$$

(i) What value should the constant α have, for this to be a legitimate probability density function?

(ii) Sketch the density f.

(iii) If no customers arrive in a five-minute window, Peter gets bored. Find the probability that $T > 5$.

(b) Peter also sells boxes of pies to his local university. Orders for boxes of pies arrive randomly, at a steady rate of two boxes every three days.

(i) Which distribution can we use to model the number of boxes of pies ordered in one day?

(ii) Peter gets overwhelmed if more than two boxes are ordered on the same day. What is the probability that he receives too many orders on a given day?

(c) At the end of the day, Peter runs a special offer: for a discount price, customers can receive a pie that has been randomly chosen among all the pies left in the shop.

One day, 20% of the remaining pies have apple filling, 30% have steak and ale filling, and the remaining 50% contain ham and pineapple. Three-quarters of the apple pies are large, along with half of the steak and ale pies and two-fifths of the ham and pineapple pies.

A customer asks Peter for a "lucky dip", and receives a large pie. What is the probability that it has apple filling?