Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH1597-WE01

Title:
Probability I

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

Q1 (a) A deck of 12 cards is made up of the Jack, Queen, and King of each of the four suits ($\spadesuit, \heartsuit, \diamondsuit, \clubsuit$). The deck is well shuffled, and dealt randomly among Alice, Bob, and yourself, so each player has a hand of 4 cards. Your hand is J$\diamondsuit$, J$\spadesuit$, K$\heartsuit$, and K$\clubsuit$. What is the probability that Alice has the remaining two cards of suit $\spadesuit$?

(b) A bag contains 3 six-sided dice, 2 eight-sided dice, and one 10-sided die; an n-sided die has sides labelled $1, 2, \ldots, n$, and all dice are assumed to be fair. A die is drawn from the bag at random and rolled. What is the expected score on the die?

(c) Suppose that X is a Poisson random variable with parameter $\lambda > 0$, so that $\mathbb{P}(X = x) = \mathrm{e}^{-\lambda}\lambda^x/x!$ for $x = 0, 1, 2, \ldots$. Calculate $\mathbb{E}[\frac{X}{1+X}]$.

Q2 (a) In a certain mail order company, Alan and Betty work on the telephones, while Chris and Delia work in the warehouse. At a particular time of day a call for an urgent order comes in. The order will be fulfilled if and only if there's at least one employee available on the telephones and both employees are available in the warehouse. At any given time, the events that each of the four staff are available to fulfil an order are independent: for each of Alan and Betty, the probability they are available is $2/3$; for each of Chris and Delia, the probability they are available is $3/4$. If a call comes in when both Alan and Betty are available, Betty (who is quicker on the draw) takes the call with probability $3/4$, otherwise it goes to Alan. Given that the urgent order was fulfilled, what is the probability that Alan took the call?

(b) A gene on an homologous chromosome in mice has alleles A, a that determine the colour of their coats. Genotypes AA and Aa give a black coat, while aa mice are brown. An Aa mouse is mated with an AA mouse, and one of their offspring, Morris, is mated with Mavis, a brown mouse. Find

(i) the probability that the first of their offspring is black;

(ii) the probability that their third offspring is black, given that the first two are black.

Q3 Discrete random variables X and Y have joint probability mass function given in terms of a parameter $\alpha \in [0, \frac{1}{10}]$ by:

$p_{X,Y}(x, y)$	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$y = 0$	0	2α	2α	α
$y = 1$	α	0	0	$\frac{1}{2} - \alpha$
$y = 2$	$\frac{1}{10} - \alpha$	$\frac{1}{5} - 2\alpha$	$\frac{1}{5} - 2\alpha$	0

(a) Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, and $\mathbb{C}\mathrm{ov}(X, Y)$. Your answers may depend on α.

(b) Calculate $\mathbb{P}(Y = y \mid X = 0)$ for $y \in \{0, 1, 2\}$. Your answers may depend on α.

(c) Give all (if any) values of α for which $\mathbb{C}\mathrm{ov}(X, Y) = 0$. Give all (if any) values of α for which X and Y are independent. Explain.

Q4 Jointly continuous random variables X, Y have joint probability density function

$$f(x,y) = \begin{cases} \theta \dfrac{y}{x^2} \mathrm{e}^{-y} & \text{for } 1 \le x \le y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the value of the constant θ.

(b) Find $\mathbb{P}(2 \le Y \le 2X)$.

(c) Calculate the marginal probability density function f_Y of Y.

(d) For all $x \in \mathbb{R}$ and all $y > 1$, calculate the value of the conditional density function $f_{X|Y}(x \mid y)$ of X given $Y = y$. By evaluating an appropriate integral against the conditional density, compute $\mathbb{E}(X \mid Y = y)$ as a function of $y > 1$.

Q5 Let Y be a continuous random variable with probability density function

$$f(y) = k\mathrm{e}^{-\beta|y|}, \text{ for } y \in \mathbb{R},$$

where $\beta > 0$ is a constant.

(a) Find the value of k in terms of β.

(b) Find the moment generating function $M_Y(t) = \mathbb{E}(\mathrm{e}^{tY})$ for all values of $t \in \mathbb{R}$.

(c) Use M_Y to find $\mathbb{E}(Y)$ and $\mathbb{V}\mathrm{ar}(Y)$.

