

EXAMINATION PAPER

Examination Session: May/June

2022

Year:

Exam Code:

MATH2011-WE01

Title:

Complex Analysis II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.		
	Students must use the mathematics specific answer book.		

Revision:

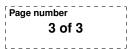
SECTION A

- **Q1** Let $A = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im}(z) \neq 0\}.$
 - (a) Show that A is an open subset of \mathbb{C} .
 - (b) Find the boundary ∂A and the exterior A^e of A.
 - (c) Is it true that for every $x \in \partial A$ and every open $U \subset \mathbb{C}$ containing x we have $U \cap A \neq \emptyset$ and $U \cap A^e \neq \emptyset$? Justify your answer.
- **Q2** Consider the smooth curve $\gamma : [0, 4\pi] \to \mathbb{C}$ defined by $\gamma(t) = t \exp(it)$.
 - 2.1 Draw the curve in the complex plane.
 - 2.2 State the Complex Fundamental Theorem of Calculus.
 - 2.3 Calculate the two integrals

$$\int_{\gamma} \sin(z) \, dz, \quad \int_{\gamma} |z| \, dz.$$

- **Q3 3.1** State what it means for a real-valued function defined on a domain $D \subseteq \mathbb{C}$ to be *harmonic*. Let f(z) = u(x, y) + iv(x, y) be a holomorphic function. Show that its real part u(x, y) is a harmonic function. (You may assume that the partial derivatives of all orders of u and v exist and are continuous).
 - **3.2** Show that the function $u(x, y) = x^3 3xy^2 + 4y$ is harmonic and find a real-valued function v(x, y) such that f(z) = u(x, y) + iv(x, y) is holomorphic.
- **Q4** Using the substitution $z = e^{it}$ or otherwise, evaluate the integral

$$\int_0^{2\pi} \frac{\cos t}{17 - 8\cos t} \, dt \, dt$$



SECTION B

- **Q5** 5.1 Let $U = \{z \in \mathbb{C} : \operatorname{Re}(z) < 0\}$ and $V = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that exp is conformal in \mathbb{C} and satisfies f(U) = V. Is exp : $U \to V$ a biholomorphism?
 - **5.2** Using part **5.1** or otherwise, find a conformal map from $\{z \in \mathbb{C} : |z| < 1\}$ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$.
- **Q6** For $n \in \mathbb{N}$ consider $f_n : \mathbb{C} \to \mathbb{C}$ given by

$$f_n(z) = \begin{cases} \frac{nz+1}{z+in}, & z \in \mathbb{C} \setminus \{-in\}, \\ 0, & z = -in. \end{cases}$$

- **6.1** Show that $(f_n)_{n \in \mathbb{N}}$ is locally uniformly convergent on \mathbb{C} . What is the limit function?
- **6.2** Find the image of $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ under f_n .
- **Q7** Let f(z) be a non-constant complex-valued function that is holomorphic on the set $\{z \in \mathbb{C} : |z| \le 1\}$ and has the property that |f(z)| < 1 whenever |z| = 1.
 - **7.1** Show that for all z in the set $\{z \in \mathbb{C} : |z| \le 1\}$ we have

$$1 - zf(z) \neq 0.$$

- **7.2** State Rouché's Theorem and use it to show that there is a *unique* complex number ω in the set $\{z \in \mathbb{C} : |z| < 1\}$ such that $f(\omega) = \omega$. Explain why this ω must also satisfy $f'(\omega) \neq 1$.
- 7.3 Hence, or otherwise, evaluate the integral

$$I = \int_{|z|=1} \frac{1 - zf(z)}{f(z) - z} \, dz \,,$$

and show that $f'(\omega) = \frac{2\pi i(1-\omega^2)+I}{I}$, where ω is as in part 7.2.

Q8 8.1 For any real $R > \sqrt{3}$ consider the 'D-shaped' contour $\Gamma_R = L_R + C_R$ consisting of the straight line L_R joining -R to R on the real axis and the semi-circular arc C_R in the upper half-plane rejoining R to -R. Evaluate the integral

$$\int_{\Gamma_R} \frac{z^2}{z^4 + 5z^2 + 6} \, dz.$$

8.2 Hence, or otherwise, evaluate the real integral

$$\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 6} \, dx.$$