Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH2011-WE01

Title:
Complex Analysis II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Let $A = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im}(z) \neq 0\}$.

(a) Show that A is an open subset of $\mathbb{C}$.

(b) Find the boundary ∂A and the exterior A^e of A.

(c) Is it true that for every $x \in \partial A$ and every open $U \subset \mathbb{C}$ containing x we have $U \cap A \neq \emptyset$ and $U \cap A^e \neq \emptyset$? Justify your answer.

Q2 Consider the smooth curve $\gamma : [0, 4\pi] \to \mathbb{C}$ defined by $\gamma(t) = t\exp(it)$.

2.1 Draw the curve in the complex plane.

2.2 State the Complex Fundamental Theorem of Calculus.

2.3 Calculate the two integrals

$$\int_\gamma \sin(z)\, dz, \qquad \int_\gamma |z|\, dz.$$

Q3 **3.1** State what it means for a real-valued function defined on a domain $D \subseteq \mathbb{C}$ to be *harmonic*. Let $f(z) = u(x,y) + iv(x,y)$ be a holomorphic function. Show that its real part $u(x,y)$ is a harmonic function. (You may assume that the partial derivatives of all orders of u and v exist and are continuous).

3.2 Show that the function $u(x,y) = x^3 - 3xy^2 + 4y$ is harmonic and find a real-valued function $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is holomorphic.

Q4 Using the substitution $z = e^{it}$ or otherwise, evaluate the integral

$$\int_0^{2\pi} \frac{\cos t}{17 - 8\cos t}\, dt\,.$$

SECTION B

Q5 **5.1** Let $U = \{z \in \mathbb{C} : \operatorname{Re}(z) < 0\}$ and $V = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that exp is conformal in $\mathbb{C}$ and satisfies $f(U) = V$. Is $\exp : U \to V$ a biholomorphism?

5.2 Using part **5.1** or otherwise, find a conformal map from $\{z \in \mathbb{C} : |z| < 1\}$ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$.

Q6 For $n \in \mathbb{N}$ consider $f_n : \mathbb{C} \to \mathbb{C}$ given by

$$f_n(z) = \begin{cases} \frac{nz+1}{z+in}, & z \in \mathbb{C}\setminus\{-in\}, \\ 0, & z = -in. \end{cases}$$

6.1 Show that $(f_n)_{n\in\mathbb{N}}$ is locally uniformly convergent on $\mathbb{C}$. What is the limit function?

6.2 Find the image of $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ under f_n.

Q7 Let $f(z)$ be a non-constant complex-valued function that is holomorphic on the set $\{z \in \mathbb{C} : |z| \leq 1\}$ and has the property that $|f(z)| < 1$ whenever $|z| = 1$.

7.1 Show that for all z in the set $\{z \in \mathbb{C} : |z| \leq 1\}$ we have

$$1 - zf(z) \neq 0.$$

7.2 State Rouché's Theorem and use it to show that there is a *unique* complex number ω in the set $\{z \in \mathbb{C} : |z| < 1\}$ such that $f(\omega) = \omega$. Explain why this ω must also satisfy $f'(\omega) \neq 1$.

7.3 Hence, or otherwise, evaluate the integral

$$I = \int_{|z|=1} \frac{1 - zf(z)}{f(z) - z}\, dz\,,$$

and show that $f'(\omega) = \dfrac{2\pi i(1 - \omega^2) + I}{I}$, where ω is as in part **7.2**.

Q8 **8.1** For any real $R > \sqrt{3}$ consider the 'D-shaped' contour $\Gamma_R = L_R + C_R$ consisting of the straight line L_R joining $-R$ to R on the real axis and the semi-circular arc C_R in the upper half-plane rejoining R to $-R$. Evaluate the integral

$$\int_{\Gamma_R} \frac{z^2}{z^4 + 5z^2 + 6}\, dz.$$

8.2 Hence, or otherwise, evaluate the real integral

$$\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 6}\, dx.$$