Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH2031-WE01

Title:
Analysis in Many Variables II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 **1.1** Use index notation to show that $\boldsymbol{\nabla} \times f\boldsymbol{V} = \boldsymbol{\nabla} f \times \boldsymbol{V} + f\boldsymbol{\nabla} \times \boldsymbol{V}$, where f is a scalar field on $\mathbb{R}^3$, and $\boldsymbol{V}$ is a vector field on $\mathbb{R}^3$.

1.2 Let $\boldsymbol{V} = \boldsymbol{a} \times \boldsymbol{x}$, where $\boldsymbol{a} \in \mathbb{R}^3$ is a constant vector, and $\boldsymbol{x} \in \mathbb{R}^3$ is the position vector in $\mathbb{R}^n$. Use index notation to show that $\boldsymbol{\nabla} \times \boldsymbol{V} = 2\boldsymbol{a}$.

1.3 Calculate $\boldsymbol{\nabla} \times r\boldsymbol{V}$, where $\boldsymbol{V} = \boldsymbol{a} \times \boldsymbol{x}$ as in the previous part of the question, and $r = |\boldsymbol{x}|$.

Q2 **2.1** State the conditions necessary for the line integral of a vector field $\boldsymbol{F}$ from x_0 to x_1 to be path independent.

2.2 Calculate $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{x}$, where $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field given as

$$\boldsymbol{F}(x, y, z) = (x^2 - y, yz - x, y^2/2),$$

and $\gamma : \mathbb{R} \to \mathbb{R}^3$ is the path parameterised in terms of t for $0 \leq t \leq 1$, as

$$\gamma(t) = \left(\sin^2\left(\frac{t\pi}{2}\right), \frac{\log(1+t)^2}{\log 2}, 3t\exp(1-t)\right).$$

Q3 The components of a vector field $\mathbf{A}$ in spherical polar coordinates (r, θ, ϕ) are given in terms of a function f by

$$A_r = \frac{1}{r^2 \sin(\theta)}\, \partial_\theta f, \qquad A_\theta = -\frac{1}{r\sin(\theta)}\, \partial_r f, \qquad A_\phi = 0.$$

Give an explicit expression for the differential operator D defined by

$$Df := -r\sin(\theta)\, (\boldsymbol{\nabla} \times \mathbf{A})_\phi$$

Note: In terms of spherical polar coordinates, the Cartesian coordinates are given by

$$x = r\sin(\theta)\, \cos(\phi), \quad y = r\sin(\theta)\, \sin(\phi), \quad z = r\cos(\theta),$$

and you might find the following information useful:

$$\partial_r \mathbf{e}_r = \partial_r \mathbf{e}_\theta = \partial_r \mathbf{e}_\phi = \mathbf{0},\ \partial_\theta \mathbf{e}_r = \mathbf{e}_\theta,\ \partial_\theta \mathbf{e}_\theta = -\mathbf{e}_r,\ \partial_\theta \mathbf{e}_\phi = \mathbf{0},$$
$$\partial_\phi \mathbf{e}_r = \sin(\theta)\mathbf{e}_\phi,\ \partial_\phi \mathbf{e}_\theta = \cos(\theta)\mathbf{e}_\phi,\ \partial_\phi \mathbf{e}_\phi = -(\sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta).$$

Q4 **4.1** Consider the first order differential equation

$$(\sin(x))^2\, T' = 0, \quad T \in \mathcal{D}'(\mathbb{R}).$$

What is its most general solution in the sense of distributions? Justify your answer fully by proving that your candidate solution satisfies the differential equation in the sense of distributions.

4.2 Calculate the distributional derivative of the piecewise smooth function f defined on $(-3\pi, 2\pi)$ by,

$$f(x) = \begin{cases} \cos(x) & \text{for } -6\pi < 2x < -\pi, \\ \frac{2}{\pi}x - \frac{1}{2} & \text{for } -\pi \le 2x < \pi, \\ \sin(\frac{\pi}{2} - x) & \text{for } \pi \le 2x < 4\pi. \end{cases}$$

Note: Giving the final answer only will not attract full marks. You must detail and justify the steps you took to arrive at the final expression.

SECTION B

Q5 **5.1** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar field on $\mathbb{R}^n$. Define what it means for the limit of $f(\boldsymbol{x})$ as $\boldsymbol{x}$ tends to $\boldsymbol{a}$ to be L.

5.2 Define what it means for f to be continuous at $\boldsymbol{a}$.

5.3 Let f be a scalar field on $\mathbb{R}^2$ given by

$$f(x, y) = \begin{cases} xy(x^2 - y^2)/(x^2 + y^2) & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

Show that f is continuous on $\mathbb{R}^2$, stating any results that you use.

5.4 Is f differentiable at the origin?

5.5 Show that, at the origin,

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}.$$

Q6 **6.1** Let $\boldsymbol{V} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field on $\mathbb{R}^3$ given as

$$\boldsymbol{V}(r,\theta,\phi) = (r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)),$$

which describes the transformation from spherical polar coordinates back to Cartesian coordinates, and let $U = \{(r,\theta,\phi) \in \mathbb{R}^3 | \; 0 < r, 0 < \theta < \pi, 0 \le \phi < 2\pi\}$. Show that $\boldsymbol{V}$ describes an orientation preserving diffeomorphism from U to $\boldsymbol{V}(U) = \mathbb{R}^3 \setminus \{(0,0,z) \mid z \in \mathbb{R}\}$.

6.2 Use the divergence theorem to calculate $\int_S \boldsymbol{F} \cdot d\boldsymbol{A}$, where S is the sphere of radius two centred on the origin with outwards pointing normal, and $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field given as

$$\boldsymbol{F}(x,y,z) = (x^3, y^3, x^2 + y^2).$$

6.3 Let M be the mushroom-shaped surface with outward-pointing normal, formed by joining the hemisphere of radius 2 whose base lies in the x,y-plane centred on the origin, to the cylinder of radius r and height 1 whose axis lies along the z-axis and whose ends lie in the planes $z = 0$ and $z = -1$. Find r such that

$$\int_M \boldsymbol{F} \cdot d\boldsymbol{A} = \int_S \boldsymbol{F} \cdot d\boldsymbol{A},$$

where S and $\boldsymbol{F}$ are as in the previous part of the question.

Q7 **7.1** Calculate the Green's function $G(t)$ for the second order linear differential operator

$$L = \frac{d^2}{dt^2} + a\frac{d}{dt} + b, \qquad a, b \in \mathbb{R},$$

where $G(t) = 0$ for $t < 0$. Distinguish between the three cases $a^2 < 4b$, $a^2 = 4b$ and $a^2 > 4b$.

Hint: you may want to try the ansatz $G(t) = g(t)\Theta(t)$ in the equation $LG = \delta$, where Θ is the Heaviside function interpreted as a regular distribution. This will give you a differential equation for $g(t)$ as well as an initial value for g and an initial value for g', which cannot depend on the parameters a, b and which you will need to proceed.

7.2 Use your results in part 7.1 to solve the following inhomogeneous problem on $\mathbb{R}$,

$$u''(t) + u'(t) + u(t) = f(t), \qquad u(t), u'(t) \to 0 \text{ as } t \to -\infty,$$

where $f(t) = \Theta(t) - \Theta(t-1)$.

Q8 Use an eigenfunction expansion of the function $u(x)$ to solve the following boundary value problem,

$$-e^{-x}\frac{d}{dx}\left(e^x \frac{du}{dx}\right) - \frac{1}{4}u = -e^{-x/2}, \quad u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} + \frac{1}{2}u(1) = 0,$$

on $(0,1)$.