EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH2051-WE01

Title:
Numerical Analysis II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

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SECTION A

Q1 (a) Define what is meant by floating-point numbers having a *finite precision* and a *finite range*.

(b) Let $x_0 = 1$ and $x_{n+1} = x_n/2$. If one was to compute x_{2000} in Python (using the usual `float`), what would one get? How does this relate to part (a)?

(c) One seeks to approximate $f'(x) \simeq [f(x+h) - f(x)]/h$, taking $h_k = 2^{-k}$ for $k = 10, \cdots, 50$. Which value of k (roughly) would give the most accurate approximation? Argue briefly why.

Q2 Consider a one dimensional root finding problem $f(x) = 0$ on a domain $[a, b]$, such that there is some solution $d \in [a, b]$.

(a) Sketch a function $f(x)$ for which the bisection method will converge using the pair (a, b) as the input range, but for which there is some $c \in [a, b]$ for which the Newton iterative method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

will fail to find the root d.

(b) Sketch a function $f(x)$ for which the bisection method will fail to converge using (a, b) as the input range, but for which there is some $d \in [a, b]$ for which the Newton iterative method does locate the root.

(c) Consider the fixed point iteration $x_{k+1} = \tan^{-1}(x_k)$ applied to $f(x) = 0$ with $f(x) = x - \tan^{-1}(x)$. Demonstrate that the contraction mapping theorem is not satisfied. Does this mean the iteration will definitely not converge?

Q3 (a) Compute the LU decomposition for

$$A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 3 & 11 & 11 & 13 \\ 2 & 16 & 30 & 15 \\ 4 & 14 & 15 & 19 \end{pmatrix}.$$

(b) Prove that the product of two $n \times n$ unit lower triangular matrices is again unit lower triangular.

Q4 Recall the Chebyshev polynomials $T_k(x) = \cos(k \cos^{-1} x)$ for $k \in \{0, 1, \cdots\}$.

(a) Show that the Chebyshev polynomials are orthogonal with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)(1 - x^2)^{-1/2}\, \mathrm{d}x.$$

(b) We would like to expand the function $f(x) = |x|$ in terms of Chebyshev polynomials,

$$p_f(x) = \sum_{k=0}^{\infty} c_k T_k(x)$$

such that, when truncating the sum at any finite $N > 0$, the L^2 error with respect to the above inner product is minimised among any polynomials of degree N. Compute explicitly the coefficients c_k. Is the infinite series likely to converge pointwise (i.e. for every $x \in (-1, 1)$)? Why?

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SECTION B

Q5 (a) Prove the following theorem. Given data $\{x_j\}_{j=0}^{n}$ and $f : \{x_j\} \to \mathbb{R}$, there exists a unique $p \in \mathcal{P}_n$ that interpolates the data, that is:

$$p(x_j) = f(x_j) \qquad \text{for } j \in \{0, \cdots, n\}.$$

(b) Construct the Newton polynomial $p_n(x)$ for the function $f(x) = \mathrm{e}^{-x^2}\cos(3\pi x)$ using the values $f(0), f(1/3), f(2/3)$ and $f(1)$. You should state the coefficients of the polynomial exactly.

(c) Consider a function f for which $f(-1)$, $f(0)$ and $f(1)$ are given. One would like to construct a cubic $p(x)$ with those data plus $f'(a)$ for some $a \in [-1, 1]$. What values of a can one take that give a unique cubic $p(x)$? Justify your answer.

Q6 Construct an explicit finite difference scheme to solve the following boundary problem for the function $u(x, t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x}, \qquad \frac{\partial u}{\partial x} = 0 \text{ on } x = 0, L,$$

for some arbitrary initial datum $u(x, 0) = g(x)$ satisfying the boundary conditions. The scheme should be second order in the spatial coordinate x and first order in t. Quote fully any result from class that you use.

Q7 (a) State the properties of a *norm*.

(b) Define what is meant by an *induced norm*.

(c) Starting from your definition, prove that every induced norm satisfies the properties of a norm.

(d) Defining the vector ∞-norm and the matrix row-sum norm, prove that the former induces the latter.

(e) Let

$$A = \begin{pmatrix} 2 & -5 & 1 \\ 1 & 3 & -4 \\ 7 & 1 & 0 \end{pmatrix}.$$

What can you say about the claim that $\|A^{100}\|_\infty \simeq 10^{100}$, to within 10%? Justify your answer.

Q8 (a) Define what is meant by *degree of exactness* of a numerical quadrature formula.

(b) Given the nodes $\{x_0, \cdots, x_n\} \subset [-1, 1]$ such that $x_{n-k} = -x_k$ for $k \in \{0, \cdots, n\}$, prove that the coefficients $\{\rho_k\}$ of the interpolatory quadratures I_n in $[-1, 1]$ with these nodes satisfy $\rho_{n-k} = \rho_k$.

(c) Construct explicitly an interpolatory quadrature in $[-1, 1]$ with nodes $\{x_0 = -1, x_1, x_2, x_3 = 1\}$ with $\{x_1, x_2\}$ and the weights $\{\rho_k\}$ to be determined so that the quadrature is of the highest possible order. What is the order of your quadrature?

(d) Why would one go through the trouble to construct such a quadrature?

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