

EXAMINATION PAPER

Examination Session: May/June

2022

Year:

Exam Code:

MATH2051-WE01

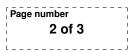
Title:

Numerical Analysis II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85
		series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.		
	Students must use the mathematics specific answer book.		

Revision:



SECTION A

- **Q1** (a) Define what is meant by floating-point numbers having a *finite precision* and a *finite range*.
 - (b) Let $x_0 = 1$ and $x_{n+1} = x_n/2$. If one was to compute x_{2000} in Python (using the usual float), what would one get? How does this relate to part (a)?
 - (c) One seeks to approximate $f'(x) \simeq [f(x+h) f(x)]/h$, taking $h_k = 2^{-k}$ for $k = 10, \dots, 50$. Which value of k (roughly) would give the most accurate approximation? Argue briefly why.
- **Q2** Consider a one dimensional root finding problem f(x) = 0 on a domain [a, b], such that there is some solution $d \in [a, b]$.
 - (a) Sketch a function f(x) for which the bisection method will converge using the pair (a, b) as the input range, but for which there is some $c \in [a, b]$ for which the Newton iterative method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

will fail to find the root d.

- (b) Sketch a function f(x) for which the bisection method will fail to converge using (a, b) as the input range, but for which there is some $d \in [a, b]$ for which the Newton iterative method does locate the root.
- (c) Consider the fixed point iteration $x_{k+1} = \tan^{-1}(x_k)$ applied to f(x) = 0 with $f(x) = x \tan^{-1}(x)$. Demonstrate that the contraction mapping theorem is not satisfied. Does this mean the iteration will definitely not converge?
- **Q3** (a) Compute the LU decomposition for

$$A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 3 & 11 & 11 & 13 \\ 2 & 16 & 30 & 15 \\ 4 & 14 & 15 & 19 \end{pmatrix}.$$

- (b) Prove that the product of two $n \times n$ unit lower triangular matrices is again unit lower triangular.
- **Q4** Recall the Chebyshev polynomials $T_k(x) = \cos(k \cos^{-1} x)$ for $k \in \{0, 1, \dots\}$.
 - (a) Show that the Chebyshev polynomials are orthogonal with respect to the inner product

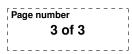
$$(f,g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{-1/2} dx.$$

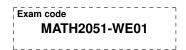
(b) We would like to expand the function f(x) = |x| in terms of Chebyshev polynomials,

$$p_f(x) = \sum_{k=0}^{\infty} c_k T_k(x)$$

such that, when truncating the sum at any finite N > 0, the L^2 error with respect to the above inner product is minimised among any polynomials of degree N. Compute explicitly the coefficients c_k . Is the infinite series likely to converge pointwise (i.e. for every $x \in (-1, 1)$)? Why?

CONTINUED





SECTION B

Q5 (a) Prove the following theorem. Given data $\{x_j\}_{j=0}^n$ and $f : \{x_j\} \to \mathbb{R}$, there exists a unique $p \in \mathcal{P}_n$ that interpolates the data, that is:

$$p(x_j) = f(x_j)$$
 for $j \in \{0, \dots, n\}$.

- (b) Construct the Newton polynomial $p_n(x)$ for the function $f(x) = e^{-x^2} \cos(3\pi x)$ using the values f(0), f(1/3), f(2/3) and f(1). You should state the coefficients of the polynomial exactly.
- (c) Consider a function f for which f(-1), f(0) and f(1) are given. One would like to construct a cubic p(x) with those data plus f'(a) for some $a \in [-1, 1]$. What values of a can one take that give a unique cubic p(x)? Justify your answer.
- **Q6** Construct an explicit finite difference scheme to solve the following boundary problem for the function u(x, t):

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} - u \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial x} = 0 \text{ on } x = 0, L,$$

for some arbitrary initial datum u(x, 0) = g(x) satisfying the boundary conditions. The scheme should be second order in the spatial coordinate x and first order in t. Quote fully any result from class that you use.

- **Q7** (a) State the properties of a *norm*.
 - (b) Define what is meant by an *induced norm*.
 - (c) Starting from your definition, prove that every induced norm satisfies the properties of a norm.
 - (d) Defining the vector ∞ -norm and the matrix row-sum norm, prove that the former induces the latter.
 - (e) Let

$$A = \left(\begin{array}{rrrr} 2 & -5 & 1 \\ 1 & 3 & -4 \\ 7 & 1 & 0 \end{array}\right).$$

What can you say about the claim that $||A^{100}||_{\infty} \simeq 10^{100}$, to within 10%? Justify your answer.

- Q8 (a) Define what is meant by *degree of exactness* of a numerical quadrature formula.
 - (b) Given the nodes $\{x_0, \dots, x_n\} \subset [-1, 1]$ such that $x_{n-k} = -x_k$ for $k \in \{0, \dots, n\}$, prove that the coefficients $\{\rho_k\}$ of the interpolatory quadratures I_n in [-1, 1] with these nodes satisfy $\rho_{n-k} = \rho_k$.
 - (c) Construct explicitly an interpolatory quadrature in [-1, 1] with nodes $\{x_0 = -1, x_1, x_2, x_3 = 1\}$ with $\{x_1, x_2\}$ and the weights $\{\rho_k\}$ to be determined so that the quadrature is of the highest possible order. What is the order of your quadrature?
 - (d) Why would one go through the trouble to construct such a quadrature?