Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH2071-WE01

Title: Mathematical Physics II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Consider a Lagrangian of the form

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \cos(q_1 + q_2)\,.$$

1.1 Write down the equations of motion for the system. You do not need to solve them.

1.2 Show that $q_1(t) = q_2(t) = 0$ is a solution of the equations of motion.

1.3 Find an approximate Lagrangian L_{app} describing small perturbations around $q_1(t) = q_2(t) = 0$.

1.4 Find the general solution of the equations of motion derived from L_{app}.

Q2 **2.1** The expression for the energy of a system with Lagrangian L is given by

$$E = \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L\,.$$

Show that if L does not depend explicitly on time, then E is conserved along solutions of the equations of motion.

2.2 Consider a theory with a Lagrangian of the form

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{4}q_1^3\dot{q}_2^4 - f(q_1, q_2)\,,$$

with f an arbitrary function of two arguments. Construct the canonical momenta p_1 and p_2 associated to q_1 and q_2.

2.3 Find the Hamiltonian for the system.

Q3 A quantum mechanical system is, at $t = 0$, prepared in a state described by the wave function

$$\psi(t = 0, x) = C\left(\frac{1}{\sqrt{2}}\psi_{E=1}(x) + e^{i\beta}\psi_{E=2}(x)\right)\,,$$

where C and β are real constants. The functions $\psi_{E=1}$ and $\psi_{E=2}$ are normalised energy-eigenfunctions of the system, with eigenvalues as indicated.

3.1 Determine the constant C. Is it possible to observe the overall phase factor of this constant? Motivate your answer.

3.2 An energy measurement is made. What are the possible outcomes, and what are the probabilities of those outcomes?

3.3 Is it possible to observe the value of the phase β? Motivate your answer.

Q4 Consider a potential for a one-dimensional quantum particle, corresponding to a uniform constant force,

$$V(x) = -Fx\,,$$

where F is a real constant.

4.1 Write down the time-independent Schrödinger equation *in momentum space* for the wave function $\tilde{\psi}(p)$, including the potential given above.

4.2 Find the solution $\tilde{\psi}(p)$ up to a normalisation constant.

SECTION B

Q5 Consider a system where the Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ depends on the positions $\boldsymbol{q} = (q_1, \ldots, q_n)$, velocities $\dot{\boldsymbol{q}} = (\dot{q}_1, \ldots, \dot{q}_n)$ and accelerations $\ddot{\boldsymbol{q}} = (\ddot{q}_1, \ldots, \ddot{q}_n)$. In this case the n Euler-Lagrange equations of motion are

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0$$

for $i \in \{1, \ldots, n\}$.

We assume that the transformation $q_i \to q_i + \epsilon a_i(\boldsymbol{q})$ (with n independent generators $a_1, \ldots, a_n$) leaves the Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ invariant to first order in ϵ.

5.1 Assume first that L does not depend on the accelerations $\ddot{q}_i$. Show that in this case the Noether charge

$$Q = \sum_{i=1}^{n} a_i \frac{\partial L}{\partial \dot{q}_i}$$

is conserved.

5.2 Coming back to the more general case where $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ does depend on the accelerations $\ddot{q}_i$, find a conserved charge Q associated to the transformation generated by the a_i. [*Hint:* Try to write the variation of L to first order in ϵ as a total time derivative.]

5.3 As an example, find the explicit form for Q when the Lagrangian is

$$L_2 := -\frac{\alpha}{2}(\ddot{q}_1^2 + \ddot{q}_2^2) + \frac{\beta}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{\gamma}{2}(q_1^2 + q_2^2)$$

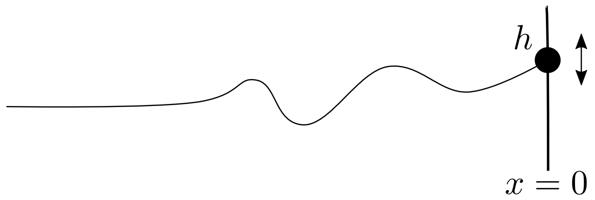
and the transformation is a rotation around the origin in the (q_1, q_2) plane.

5.4 Verify by taking the time derivative that the explicit Q that you just found is conserved along solutions of the equations of motion for the Lagrangian L_2.

Q6 We describe a string oscillating in one dimension by a field $u(x,t)$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}mu^2$$

with $u_t := \frac{\partial u}{\partial t}$ and $u_x := \frac{\partial u}{\partial x}$. The string extends from $x = -\infty$ to $x = 0$, where it ends on a bead of mass h that is constrained to move on the vertical line $x = 0$. The bead can slide without friction along $x = 0$, and we ignore the effect of gravity.



6.1 Find the equation of motion for u valid in the region $x < 0$.

6.2 Using that the energy-momentum tensor is given by

$$T_{ij} = \frac{\partial \mathcal{L}}{\partial u_j}\frac{\partial u}{\partial x_i} - \delta_{ij}\mathcal{L}\,,$$

where $u_j := \partial u/\partial x_j$, compute the energy flux T_{tx} associated to $\mathcal{L}$.

6.3 Consider an ansatz given by

$$u(x,t) = \mathrm{Re}\left(e^{i\omega t}(e^{-ikx} + \rho e^{ikx})\right)\,,$$

where $\mathrm{Re}(z)$ indicates taking the real part of z, and ρ is a complex number, which for generic h we assume to be different from -1. Find the values of ω and ρ (as functions of k, m and h) that make this ansatz a solution of the problem. [*Hint:* Impose energy conservation at the boundary.]

6.4 Consider the $(m,h) = (0,0)$ case. Which standard boundary condition for the massless scalar does this correspond to? Similarly, which standard boundary condition do you obtain in the $(m,h) \to (0,\infty)$ limit? Show in both cases that the form of $u(x,t)$ is the expected one.

Q7 Consider a system of two one-dimensional, distinguishable particles in a simple harmonic oscillator potential of frequency ω. The coordinates of the two particles are x_1 and x_2.

7.1 Give the energy eigenfunctions for the two-particle system in terms of the energy eigenfunctions of the one-particle simple harmonic oscillator. Show that the energy eigenvalues for the two-particle wave functions take the form $E = (n_1 + n_2 + 1)\hbar\omega$ (with n_1 and n_2 integers).

7.2 Now assume that the particles are *indistinguishable*. This means that all probabilities have to remain unchanged under an exchange $x_1 \leftrightarrow x_2$, so

$$\Big|\psi(x_1, x_2, t)\Big|^2 = \Big|\psi(x_2, x_1, t)\Big|^2 .$$

Show that there are now only *two* independent wave functions $\psi(x_1, x_2, t)$ for a state with energy $E = 2\hbar\omega$ (up to an irrelevant phase factor).

7.3 The explicit form of the first two wave functions for the single particle system are given by

$$\psi_{n=0}(x) = C \exp\left(-\frac{m\omega x^2}{2\hbar}\right) , \quad \psi_{n=1}(x) = \sqrt{\frac{2m\omega}{\hbar}} x \times \psi_{n=0}(x) .$$

Rewrite the two wave functions found in **7.2** in terms of the centre of mass X and separation y,

$$X = \frac{1}{2}(x_1 + x_2) , \quad y = x_1 - x_2 .$$

7.4 Determine the probability density for the separation $P(y)$ by averaging over X, separately for the two wave functions found in **7.3**. You do *not* have to work out the overall normalisation factors.

7.5 Sketch the form of $P(y)$ for the two cases. Which of the two describes a system in which particles repel each other?

Q8 Consider the time-dependent Schrödinger equation for a single free particle. A Galilean transformation is a transformation on x and p which takes us to the coordinates of a moving observer, so in particular $x \to x' = x + vt$ (and $t = t'$).

8.1 Write down the Schrödinger equation (and include a generic potential $V(x,t)$). Transform coordinates to x', t', assuming that the wave function in primed coordinates is related to the one in unprimed coordinates by a phase factor,

$$\psi'(x',t') = \exp\Big[i f(x',t') \Big] \psi(x,t)\,,$$

where $f(x',t')$ is real. Use that $V'(x',t') = V(x,t)$. [*Hint:* First show that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\,,\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + v\frac{\partial}{\partial x'}\,.$$

and then apply this to the Schrödinger equation.]

8.2 Show that the Schrödinger equation in primed coordinates has the same form as the one in unprimed coordinates when

$$f(x',t') = \frac{mvx' - \frac{1}{2}mv^2 t'}{\hbar}\,.$$

8.3 Consider now the situation $V(x) = 0$. The momentum eigenfunction for a single particle reads

$$\psi(x,t) = e^{\frac{i}{\hbar}px - i\omega t}\,,\quad \omega = \frac{p^2}{2\hbar m}\,.$$

Transform it to $\psi'(x',t')$ and show that the result is consistent with the transformation which you expect $x' = x + vt$ to induce on the momentum.