

EXAMINATION PAPER

Examination Session: May/June

2022

Year:

Exam Code:

MATH2581-WE01

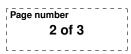
Title:

Algebra II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.		
	Students must use the mathematics specific answer book.		

Revision:



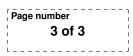
SECTION A

- **Q1 1.1** How many monic irreducible polynomials of degree 2 are there in $\mathbb{Z}/5[x]$? (Provide your reasoning.)
 - **1.2** Let R be a commutative ring, let $a, b \in R$ be such that $a^n = 0$ and $b^m = 0$ for some m, n > 0. Show that $(a - b)^k$ equals zero for some suitable $k \in \mathbb{Z}$.
- **Q2** Consider the map $\varphi : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}/30$ given by $\varphi(a + b\sqrt{-5}) = \overline{a + 5b}$ (where the bar indicates the reduction modulo 30, as usual).
 - **2.1** Show that φ is a surjective homomorphism of rings.
 - **2.2** Find the kernel of φ and show that it is a principal ideal.
- Q3 3.1 Write the following permutation

 $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 3 & 2 & 7 & 8 & 5 & 1 & 6 & 4 \end{pmatrix}$

as a product of disjoint cycles and as a product of transpositions. What is the sign of σ^{-3} ?

- **3.2** Determine all the groups that can occur as *proper* subgroups of a group of order 242, up to isomorphism. For any such subgroup you need to show that it does indeed occur as a subgroup of some group of order 242. (Carefully state any results that you use.)
- **Q4** 4.1 Determine all the conjugacy classes of D_6 . Using these, or otherwise, determine all the proper normal subgroups of D_6 . (Justify your reasoning.)
 - **4.2** List five groups of order 8, up to isomorphism. (Justify why they are not isomorphic to one another.)





SECTION B

Q5 On the set $S = \mathbf{Q}^3$ define the first binary operation + to be vector addition, and define the second binary operation * by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 + a_2 b_3 \\ a_3 b_3 \end{pmatrix}.$$

- **5.1** Show that (S, +, *) constitutes a ring with identity.
- **5.2** Does S have non-trivial zero divisors?
- **5.3** Determine the units in S, if any.

[Hint: From Linear Algebra you can assume that $(\mathbf{R}^3, +)$ (vector addition) constitutes an abelian group. You can also assume that $(\mathbf{Q}, +, \cdot)$ (usual addition and multiplication) is a ring.]

- **Q6 6.1** For a polynomial $f(x) \in \mathbf{Q}[x]$ of degree $n = \deg(f(x))$ show that $x^n f\left(\frac{1}{x}\right)$ also lies in $\mathbf{Q}[x]$. Moreover, show that, if f(x) is reducible in $\mathbf{Q}[x]$, then so is $x^n f\left(\frac{1}{x}\right)$.
 - **6.2** Using the previous part, or otherwise, show the irreducibility of the polynomial $f(x) = 2x^4 + 4x^2 + 4x + 3$ in $\mathbf{Q}[x]$. (Carefully cite any result you are using.)
 - **6.3** Writing $\overline{f}(x)$ for the reduction of f(x) in $\mathbb{Z}/7[x]$, determine the monic gcd of $\overline{g}(x) = \overline{2}x^3 + \overline{3}x^2 + \overline{3}x + \overline{1} \in \mathbb{Z}/7[x]$ and $\overline{f}(x)$, and express this gcd explicitly in terms of $\overline{f}(x)$ and $\overline{g}(x)$.
- **Q7** 7.1 Let G be a group of order p^3 for a prime number p. Determine all the possible sizes that the centre of G can have, and give an example for each of these cases when p = 2. (Carefully cite any results from lectures that you use.)
 - **7.2** List all the conjugacy classes in $G = A_5$. For each class, give its size. Choose a representative of a class of largest size and determine its stabiliser in G.
- **Q8** Let G be a group and put, for each $g \in G$,

$$Z_G(g) = \{h \in G \mid hg = gh\}.$$

- **8.1** Show that $Z_G(g)$ is a subgroup of G.
- **8.2** Is it always true that $Z_G(g)$ is normal in G?
- **8.3** Given two elements g and g' in the same conjugacy class of G, both of which are different from the identity in G, is it true that $Z_G(g)$ is always conjugate to $Z_G(g')$?
- 8.4 Determine $Z_G(g)$ for each element g of order 2, up to conjugacy, in $G = D_4$.