Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH2581-WE01

Title: Algebra II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 **1.1** How many monic irreducible polynomials of degree 2 are there in $\mathbf{Z}/5[x]$? (Provide your reasoning.)

1.2 Let R be a commutative ring, let $a,\, b \in R$ be such that $a^n = 0$ and $b^m = 0$ for some $m, n > 0$.
Show that $(a-b)^k$ equals zero for some suitable $k \in \mathbf{Z}$.

Q2 Consider the map $\varphi : \mathbf{Z}[\sqrt{-5}] \to \mathbf{Z}/30$ given by $\varphi(a + b\sqrt{-5}) = \overline{a + 5b}$ (where the bar indicates the reduction modulo 30, as usual).

2.1 Show that φ is a surjective homomorphism of rings.

2.2 Find the kernel of φ and show that it is a principal ideal.

Q3 **3.1** Write the following permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 3 & 2 & 7 & 8 & 5 & 1 & 6 & 4 \end{pmatrix}$$

as a product of disjoint cycles and as a product of transpositions. What is the sign of σ^{-3}?

3.2 Determine all the groups that can occur as *proper* subgroups of a group of order 242, up to isomorphism. For any such subgroup you need to show that it does indeed occur as a subgroup of some group of order 242. (Carefully state any results that you use.)

Q4 **4.1** Determine all the conjugacy classes of D_6. Using these, or otherwise, determine all the proper normal subgroups of D_6. (Justify your reasoning.)

4.2 List five groups of order 8, up to isomorphism. (Justify why they are not isomorphic to one another.)

SECTION B

Q5 On the set $S = \mathbf{Q}^3$ define the first binary operation $+$ to be vector addition, and define the second binary operation $*$ by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 + a_2 b_3 \\ a_3 b_3 \end{pmatrix}.$$

5.1 Show that $(S, +, *)$ constitutes a ring with identity.

5.2 Does S have non-trivial zero divisors?

5.3 Determine the units in S, if any.

[Hint: From Linear Algebra you can assume that $(\mathbf{R}^3, +)$ (vector addition) constitutes an abelian group. You can also assume that $(\mathbf{Q}, +, \cdot)$ (usual addition and multiplication) is a ring.]

Q6 **6.1** For a polynomial $f(x) \in \mathbf{Q}[x]$ of degree $n = \deg(f(x))$ show that $x^n f\left(\frac{1}{x}\right)$ also lies in $\mathbf{Q}[x]$.
Moreover, show that, if $f(x)$ is reducible in $\mathbf{Q}[x]$, then so is $x^n f\left(\frac{1}{x}\right)$.

6.2 Using the previous part, or otherwise, show the irreducibility of the polynomial $f(x) = 2x^4 + 4x^2 + 4x + 3$ in $\mathbf{Q}[x]$. (Carefully cite any result you are using.)

6.3 Writing $\overline{f}(x)$ for the reduction of $f(x)$ in $\mathbf{Z}/7[x]$, determine the monic gcd of $\overline{g}(x) = \overline{2}x^3 + \overline{3}x^2 + \overline{3}x + \overline{1} \in \mathbf{Z}/7[x]$ and $\overline{f}(x)$, and express this gcd explicitly in terms of $\overline{f}(x)$ and $\overline{g}(x)$.

Q7 **7.1** Let G be a group of order p^3 for a prime number p.
Determine all the possible sizes that the centre of G can have, and give an example for each of these cases when $p = 2$.
(Carefully cite any results from lectures that you use.)

7.2 List all the conjugacy classes in $G = A_5$. For each class, give its size. Choose a representative of a class of largest size and determine its stabiliser in G.

Q8 Let G be a group and put, for each $g \in G$,

$$Z_G(g) = \{h \in G \mid hg = gh\}.$$

8.1 Show that $Z_G(g)$ is a subgroup of G.

8.2 Is it always true that $Z_G(g)$ is normal in G?

8.3 Given two elements g and g' in the same conjugacy class of G, both of which are different from the identity in G, is it true that $Z_G(g)$ is always conjugate to $Z_G(g')$?

8.4 Determine $Z_G(g)$ for each element g of order 2, up to conjugacy, in $G = D_4$.