Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH2627-WE01

Title:
Geometric Topology II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

Unless stated otherwise, you are allowed to use any result from the lecture notes or problem sheets provided you state it correctly.

SECTION A

Q1 Given a link diagram D, we may colour it with three different colours according to the following two rules.

- Each arc is assigned exactly one colour;
- At each crossing, either all arcs have the same colour, or arcs of all three colours are used.

Every way of colouring D in such a way is referred to as a *colouring*. We define the *colouring number* of D, denoted by $\mathrm{col}(D)$, as the number of different colourings (up to relabeling) of D. For example, the colouring number of the standard diagram of the unknot is 1 as there is only one arc which—according to the rules—can only have one colour (up to relabeling).

(a) Show that the colouring number of the standard diagram of the trefoil knot is 2.

(b) Show that the colouring number is invariant under $(R1)$ and $(R2)$ moves.

(c) One can show—and you can take this for granted in the following—that the colouring number is also invariant under $(R3)$ moves. Accordingly, we may speak of the colouring number of a knot K which we may also denote by $\mathrm{col}(K)$. One can further show—and again, you can take this for granted in the following—that

$$\mathrm{col}(K_1 + K_2) = \mathrm{col}(K_1) \cdot \mathrm{col}(K_2).$$

As always, the above sum is understood to be a composition in the case of non-invertible knots.

Using the above formula, show that there is no knot K such that $T + K = U$. Here, T denotes the trefoil knot and U denotes the unknot.

Q2 (a) Give a sequence of Reidemeister moves (and unambiguously specify where each move takes place) which shows that the two link diagrams below are isotopic.

(b) Orient the components of the diagram on the left-hand side in two different ways to obtain two non-isotopic diagrams D_1 and D_2.

(c) Recall that we may apply Seifert's algorithm to oriented link diagrams. Denote by S_{D_1} and S_{D_2} the surfaces that we obtain by applying Seifert's algorithm to D_1 and D_2, respectively (with D_1 and D_2 your oriented diagrams from (b)). Are S_{D_1} and S_{D_2} equivalent?

SECTION B

Q3 In this problem, you may use without proof that the bracket polynomial is invariant under $(R2)$ and $(R3)$ moves.

(a) Give the defining relations of the Bracket polynomial.

(b) Show <u>one</u> of the following two equations (you may use both in later parts of this question)

$\Big\langle$  $\Big\rangle = -A^3 \Big\langle$ $\Big\rangle$, $\qquad$ $\Big\langle$  $\Big\rangle = -A^{-3} \Big\langle$ $\Big\rangle$.

(c) Given a Laurent polynomial $p[A]$, we denote the difference between its highest power and its lowest power by $\mathrm{span}(p)$. For example, given a polynomial

$$p[A] = A^2 - 2 - 7A^{-1} + 3A^{-4},$$

we have $\mathrm{span}(p) = 2 - (-4) = 6$. Show that $\mathrm{span}\langle D\rangle$ is an isotopy invariant.

(d) According to the above, given a knot K, we may write $\mathrm{span}\langle K\rangle$ to refer to $\mathrm{span}\langle D\rangle$ with D any diagram for K. Let T' be the mirror image of the trefoil knot. Compute $\mathrm{span}\langle T'\rangle$ and conclude that T' is not the unknot.

(e) Let $\alpha = \exp(i\pi/3)$. Show that $\langle D\rangle(\alpha)$ (the bracket polynomial evaluated at α) is an isotopy invariant. Show that $\langle D\rangle(\alpha) = 1$ for all link diagrams D.
Hint: You may find useful that $\cos(2\pi/3) = -1/2$.

Q4 (a) State the Classification Theorem for compact connected surfaces.

(b) Let S be the compact, connected, orientable surface of genus g with d open discs removed, where $g, d \geq 0$. Show that $\chi(S) = 2 - 2g - d$.

(c) Recall that a convex polyhedron is the convex hull of finitely many points in $\mathbb{R}^3$ which are not coplanar (that is, they do not lie in one plane). Each face of a convex polyhedron is a convex polygon; see the figure at the bottom of this page for an example. Given a convex polyhedron, we denote the number of k-gons among its faces by n_k (where $k \geq 3$). That is, n_3 is the number of triangle faces, n_4 is the number of quadrilateral faces etc. (e.g. in the below example, we have $n_3 = 8$ and $n_k = 0$ for all $k > 3$).
Show that for every convex polyhedron, we have

$$3n_3 + 2n_4 + n_5 \geq 12 + n_7 + 2n_8 + 3n_9 + 4n_{10} + \dots$$

When do we have equality in the above expression?
Hint: You may use without proof that for every polyhedron

$$2 = V - E + F,$$

where V is the number of vertices, E is the number of edges and F is the number of faces of the respective polyhedron.