EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH2647-WE01

Title:
Probability II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	
	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Let X be a finite random variable, $\mathsf{P}(|X| < \infty) = 1$, defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Write $\mathsf{F}_X(y)$ for its cumulative distribution function, $\mathsf{F}_X(y) = \mathsf{P}(X \leq y)$.

1.1 Carefully show that $\mathsf{P}(-m < X \leq m) \to 1$ as $m \to \infty$. Deduce that

$$\lim_{y \downarrow -\infty} \mathsf{F}_X(y) = 0 \qquad \text{and} \qquad \lim_{y \uparrow +\infty} \mathsf{F}_X(y) = 1.$$

1.2 Carefully show that $\mathsf{F}_X(\,\cdot\,)$ is right-continuous: for each real y,

$$\lim_{u \downarrow y} \mathsf{F}_X(u) = \mathsf{F}_X(y).$$

Q2 Let $([0,1], \mathcal{B}[0,1], \mathsf{P})$ be the canonical probability space, and, for real $\beta \geq 0$, let $(X_n)_{n \geq 1}$ be the sequence of random variables

$$X_n(\omega) = \begin{cases} n^\beta\,, & \omega \in (0, 1/n), \\ 0\,, & \text{otherwise.} \end{cases}$$

2.1 Find all $\beta \geq 0$ for which the monotone convergence theorem can be applied to $(X_n)_{n \geq 1}$, and for all such β, compute $\lim_{n \to \infty} \mathsf{E}(X_n)$.

2.2 Find all $\beta \geq 0$ for which the dominated convergence theorem can be applied to $(X_n)_{n \geq 1}$, and for all such β, compute $\lim_{n \to \infty} \mathsf{E}(X_n)$.

In your answer you should clearly state and carefully apply every result you use.

SECTION B

Q3 In a multiple-choice examination, consisting of an unlimited number of questions, a student chooses between one true and three false answers to each question. Assume that the student answers each question independently and uniformly at random, and let N be the number of such answers until they first answer two successive questions correctly. Denote $p_k = \mathsf{P}(N = k)$.

(a) Find p_0, p_1, p_2, and show that for all $n > 2$,

$$p_n = \frac{3}{4}p_{n-1} + \frac{3}{16}p_{n-2}\,.$$

(b) Compute the generating function $\mathsf{E}(s^N)$. Hence or otherwise, find $\mathsf{P}(N < \infty)$ and $\mathsf{E}(N)$.

Q4 Let $\big(X_n\big)_{n\ge1}$ be a sequence of random variables such that for some real γ and positive α, we have

$$\mathsf{P}\big(X_n = n^\gamma\big) = \frac{1}{n^\alpha}\,, \qquad \mathsf{P}\big(X_n = 0\big) = 1 - \frac{1}{n^\alpha}\,.$$

For each of the following statements, prove the result in the maximal region of the parameter values for which it is true and show that the claim is false for any remaining values of the parameters by providing a suitable counter-example:

4.1 As $n \to \infty$, $X_n \to 0$ in probability.

4.2 As $n \to \infty$, $X_n \to 0$ in L^{2022}.

4.3 As $n \to \infty$, $X_n \to 0$ almost surely.

In your answer you should define the relevant modes of convergence, and give a clear statement of any result you use.