Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH2707-WE01

Title: Markov Chains II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Suppose P is a stochastic matrix

$$P = \begin{bmatrix} p_{11} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & p_{42} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) What must the values of p_{11} and p_{42} be?

(b) What are the associated communicating classes?

(c) Which of the associated communicating classes are recurrent?

(d) Describe the set of stationary distributions of P.

Q2 Let $(X_n)_{n\geq 0}$ be the Markov chain that takes steps like a knight moving uniformly at random on the (unusually shaped) chess board in Figure 1, and suppose $X_0 = x$, where x is the labelled square in Figure 1. Let $T_x = \inf\{n > 0 \mid X_n = x\}$ be the first return time to x. What is $\mathbb{E}_x T_x$?

Hint: The moves a knight is possibly allowed to make are illustrated in Figure 2. The knight cannot make a move that would leave the board.

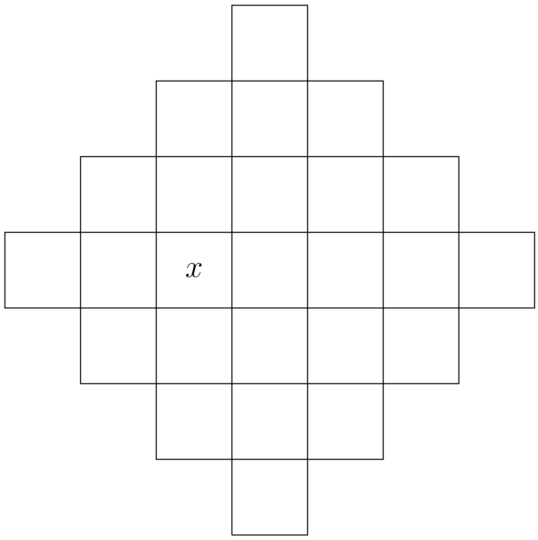

Figure 1: Unusually shaped chess board for problem **Q2**.

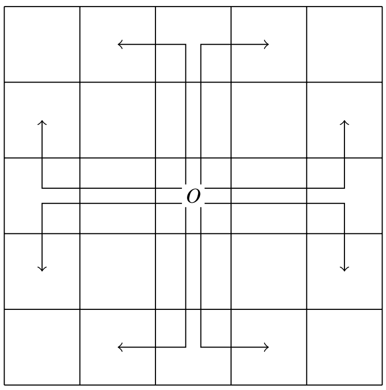

Figure 2: For problem **Q2**: the possible moves of a knight starting at the square labelled o are indicated by arrows.

SECTION B

Q3 Consider a Markov chain $(X_n)_{n\geq 0}$ on a finite state space I. Carefully prove or disprove the following statements.

3.1 All states in the same communicating class have the same period.

3.2 Suppose $(X_n)_{n\geq 0}$ is irreducible and periodic with period 2. Let P be the transition matrix of $(X_n)_{n\geq 0}$. Then -1 is an eigenvalue of P.

Q4 **4.1** Consider the set $\mathcal{P}$ of transition matrices for which the associated diagram is Figure 3, i.e., transition matrices for which all non-diagonal entries indicated by arrows in Figure 3 are positive, and non-diagonal entries are zero if there is no arrow. Any diagonal entry is also permitted to be positive. Give an example of a transition matrix $P \in \mathcal{P}$ such that the stationary distribution π of a Markov chain with transition matrix P is

$$\begin{bmatrix} \pi_A & \pi_B & \pi_C & \pi_D & \pi_E \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} & \frac{1}{5} & \frac{1}{10} & \frac{2}{5} \end{bmatrix}.$$

You must explicitly write P with each entry expressed as an explicit number.

4.2 Let $(X_n)_{n\geq 0}$ be a Markov chain on $I = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, let $0 < \tilde{p} < \frac{1}{2}$, and suppose the non-zero entries of the transition matrix P are:

- $p_{0,0} = 1 - 2\tilde{p}$,
- if $i \geq 0$, then $p_{i,i+1} = \tilde{p}$ and $p_{i+1,i} = 1 - \tilde{p}$,
- if $i \leq 0$ then $p_{i,i-1} = \tilde{p}$ and $p_{i-1,i} = 1 - \tilde{p}$.

(a) Is this Markov chain transient or recurrent? You must justify your answer.

(b) What is the long-run fraction of time this Markov chain spends at the state 0?

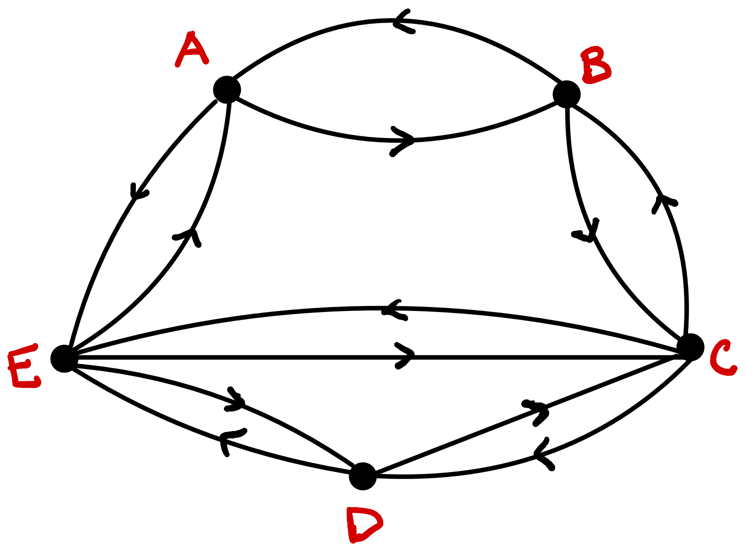

Figure 3: Figure for Problem **Q4**, part **4.1**