Durham University

EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
|---|---|---|
| May/June | 2022 | MATH3011-WE01 |

**Title:** Analysis III

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| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

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| **Revision:** | |

ED01/2022

## SECTION A

**Q1** **1.1** (i) Let $A \subseteq \mathbb{R}$. Define the outer measure of $A$, denoted $m^*(A)$.

(ii) Let $y \in \mathbb{R}$ and $B \subseteq \mathbb{R}$. Prove that

$$m^*((A - y) \cup B) \leq m^*(A) + m^*(B).$$

**1.2** Let $E \subseteq \mathbb{R}$ be measurable. Let $y \in \mathbb{R}$. Prove that the set $y + E$ is measurable and that

$$m(E) = m(y + E),$$

where $m$ denotes the Lebesgue measure.

**Q2** Let $E \subseteq \mathbb{R}$ be measurable.

**2.1** State what it means for an extended real-valued function $f : E \to \mathbb{R} \cup \{\infty\}$ to be measurable.

**2.2** By using the fact that the collection of measurable sets in $\mathbb{R}$ is an algebra, prove that any finite intersection of measurable sets is measurable.

**2.3** For a finite family $\{f_k\}_{k=1}^n$ of measurable functions $f_k : E \to \mathbb{R}$, we define the function $g : E \to \mathbb{R}$ as

$$g(x) = \min\{f_1, f_2, \ldots, f_n\}(x) = \min\{f_1(x), f_2(x), \ldots, f_n(x)\}.$$

Prove that the function $g$ is measurable.

**Q3** Let $E \subset \mathbb{R}$ be measurable and let $1 \leq p < \infty$.

**3.1** State the definition of $L^p(E)$.

**3.2** Let $f \in L^1(E) \cap L^3(E)$. Is it true that $f \in L^2(E)$? Justify your response.

**3.3** Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence of functions in $L^2(E)$ that converges to $f \in L^2(E)$. Does $(f_k)_{k\in\mathbb{N}}$ converge in measure to $f$? Justify your response.

**Q4** Let $H$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Let the norm derived from $\langle \cdot, \cdot \rangle$ be denoted by $\| \cdot \|$.

**4.1** Prove that for any $x, y \in H$,

$$\|x - y\|^2 = \|x\|^2 - 2\mathrm{Re}(\langle x, y \rangle) + \|y\|^2,$$

where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$.

**4.2** State and prove Bessel's Inequality for a Hilbert space $H$ and an orthonormal set $U \subset H$. (You may use the expression given in part **4.1**).

**4.3** State the additional assumptions that are required on $U$ in order to achieve equality.

## SECTION B

**Q5** **5.1** Let $n \in \mathbb{N}$ and $(f_n)_n$ be a sequence of functions, $f_n : \mathbb{R} \to \mathbb{R}$. State what it means that the sequence $(f_n)_n$ converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$.

**5.2** For $n \in \mathbb{N}$, let

$$g_n(x) = \frac{1}{n^{1/2}} \cdot \chi_{[n,\infty)}(x), \quad x \in \mathbb{R}.$$

(i) Prove that the sequence of functions $(g_n)_n$ converges uniformly to the function

$$g(x) = 0, \quad x \in \mathbb{R}.$$

(ii) Does Fatou's Lemma apply to the sequence of functions $(g_n)_n$? If so, then state the result of Fatou's Lemma for the sequence of functions $(g_n)_n$ given above. If not, then explain why not.

(iii) Does the Monotone Convergence Theorem apply to the sequence of functions $(g_n)_n$? If so, then state the result of the Monotone Convergence Theorem for the sequence of functions $(g_n)_n$ given above. If not, then explain why not.

**Q6** Let $E \subseteq \mathbb{R}$ be measurable.

**6.1** State what it means for a function $f : E \to \mathbb{R}$ to be integrable.

**6.2** State the Lebesgue Dominated Convergence Theorem.

**6.3** Prove the following claim: if $f : \mathbb{R} \to \mathbb{R}$ is non-negative and integrable, then

$$\lim_{n\to\infty} \int_{1-\sqrt{n}}^{1+\sqrt{n}} f = \int_{\mathbb{R}} f.$$

**6.4** Consider the sequence of functions $(f_k)_k$, $k \in \mathbb{N}$, where $f_k : \mathbb{R} \to \mathbb{R}$ is given by

$$f_k(x) = k \cdot \chi_{[0,1/k]}(x).$$

Does the Lebesgue Dominated Convergence Theorem apply to the sequence $(f_k)_k$? Give a full justification of your response.

**Q7** Let $N \in \mathbb{N}$. For $k \in \{1, \dots, N\}$, let $X_k$ be normed linear spaces with corresponding norms denoted by $\| \cdot \|_{X_k}$. Consider the space

$$Z = \bigcap_{k=1}^{N} X_k$$

where we assume that all the $X_k$ have the same operations of addition and scalar multiplication so that $Z$ is a linear space.
Define the function $\| \cdot \|_Z : Z \to \mathbb{R}$ as

$$\|w\|_Z = \sum_{k=1}^{N} \|w\|_{X_k}, \quad w \in Z.$$

**7.1** Prove that $\| \cdot \|_Z$ defines a norm on $Z$.

**7.2** Give an explicit example of $(Z, \| \cdot \|_Z)$ to show that it is not always a Banach space and justify your response briefly.

**7.3** Let $E = [0, \pi]$. Let $X_1 = L^2(E)$ where $\| \cdot \|_{X_1}$ is the usual $L^2$-norm, $X_2 = L^3(E)$ where $\| \cdot \|_{X_2}$ is the usual $L^3$-norm, and $X_3 = L^6(E)$ where $\| \cdot \|_{X_3}$ is the usual $L^6$-norm. For $k \in \mathbb{N}$, consider the functions

$$g_k(x) := \frac{(\sin(\sqrt{k}x))^3}{k^{7/6}} \cdot \chi_{[0,\pi/\sqrt{k}]}.$$

Does the sequence $(g_k)_k$ converge in $Z = L^2(E) \cap L^3(E) \cap L^6(E)$ with respect to $\| \cdot \|_Z$? Give a full justification of your response.

**Q8**

**8.1** Prove that the function

$$\langle f, g \rangle = \int_{\mathbb{R}} f\overline{g}$$

is well defined for $f, g \in L^2(\mathbb{R})$ and gives rise to an inner product on $L^2(\mathbb{R})$.

**8.2** Let the norm derived from $\langle \cdot, \cdot \rangle$ be denoted by $\| \cdot \|$. Let $E \subset \mathbb{R}$ be measurable. Let $S \subset L^2(\mathbb{R})$ be the space of functions that vanish almost everywhere in $\mathbb{R} \setminus E$. For $f \in L^2(\mathbb{R})$, prove that

$$\|f - g\| \geq \|f - \chi_E \cdot f\|$$

for all $g \in S$.