Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3021-WE01

Title:
Differential Geometry III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Let $\alpha : \mathbb{R} \to \mathbb{R}^3$, $\alpha(t) = (\sin t, \cos t, t^2)$. Calculate the curvature and the torsion of the curve α.

Q2 (a) Give the Serret-Frenet formulae for a smooth unit speed space curve $c : [a, b] \to \mathbb{R}^3$ with nowhere vanishing curvature.

(b) Assume that the curve c in (a) has constant curvature $\kappa \equiv C \neq 0$ and vanishing torsion $\tau \equiv 0$. Show that this implies $\mathbf{t}'' = -C^2\mathbf{t}$ for the unit tangent vector.

(c) Give a curve c which satisfies the conditions $\kappa \equiv 1$ and $\tau \equiv 0$. You do not need to prove that your curve has these properties.

Q3 Let $S \subset \mathbb{R}^3$ be a regular surface.

(a) Give the definition of a geodesic $c : I \to S$, where $I \subset \mathbb{R}$ is an interval.

(b) Let $c : \mathbb{R} \to S$ be a geodesic with $c'(t) \neq 0$ for all $t \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Show that $c \circ f : \mathbb{R} \to S$ is again a geodesic if and only if f is a linear function.

Q4 Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the hyperbolic plane with $E(x, y) = G(x, y) = \frac{1}{y^2}$ and $F(x, y) = 0$. Let $A \subset \mathbb{H}^2$ be the bounded domain (with respect to the hyperbolic metric) bounded by the four curves $x = -1/2$, $x = 1/2$, $x^2+y^2 = 1/2$ and $x^2+y^2 = 1$ with the four vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ and interior angles $\pi/4, \pi/4, 2\pi/3, 2\pi/3$ at these vertices (you do not need to prove this). Let $B \subset \mathbb{H}^2$ be the Euclidean rectangle with these four vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$.

(i) Derive the hyperbolic area of the domain A by applying the Gauss-Bonnet Theorem.

(ii) Calculate the hyperbolic area of the domain B by using the explicit formula for surface area.

SECTION B

Q5 Let $c : [0, 2\pi] \to \mathbb{R}^2$ be the smooth closed plane curve

$$c(t) = (x(t), y(t)) = (1 + 2\cos t)(\cos t, \sin t).$$

(a) Give the notion of a vertex of a plane curve and formulate the Four Vertex Theorem.

(b) Use $\|c'(t)\|^2 = 5 + 4\cos t$ and $x'(t)y''(t) - x''(t)y'(t) = 9 + 6\cos t$ to find the vertices of the curve c. You do not need to prove these identities.

(c) Give a reason why this result does not contradict the Four Vertex Theorem.

Q6 Let $\alpha : [a,b] \to \mathbb{R}^3$ be a simple smooth unit speed curve with nowhere vanishing curvature and associated moving frame $\mathbf{t}, \mathbf{n}, \mathbf{b} : [a,b] \to \mathbb{R}^3$.

(a) For $r > 0$ compute the coefficients E, F, G of the first fundamental form of the local parametrisation

$$\mathbf{x}(u,v) = \alpha(u) + r(\cos(v) \cdot \mathbf{n}(u) + \sin(v) \cdot \mathbf{b}(u))$$

of the canal surface $S_r \subset \mathbb{R}^3$ with radius r about α.

(b) Show that $\mathbf{N}(\mathbf{x}(u,v)) = \cos(v)\mathbf{n}(u) + \sin(v)\mathbf{b}(u)$ is a Gauss map of S_r.

(c) Compute $\mathbf{x}_{uv}(u,v)$ for the local parametrisation of S_r in (a) and the coefficient M of the second fundamental form.

(d) Assume that the curve α is contained in a plane. Decide whether $\mathbf{x}(u,v)$ from (a) is isothermal and whether it is a principal parametrisation.

Q7 Let $S \subset \mathbb{R}^3$ be a regular surface.

(a) Give the definition of an asymptotic curve $\alpha : I \to S$, where $I \subset \mathbb{R}$ is an interval.

(b) Describe the condition on a regular curve α to be an asymptotic curve in terms of the second fundamental form.

(c) Assume we have a global parametrisation $\mathbf{x} : \mathbb{R}^2 \to S$ such that the coefficients of the second fundamental form with respect to $\mathbf{x}$ are given by

$$L \equiv 1, \quad M \equiv 1, \quad N \equiv 1.$$

Show that a regular curve $\alpha : \mathbb{R} \to S$ with $\alpha(t) = \mathbf{x}(u(t), v(t))$ is an asymptotic curve if and only if there exists a constant $C \in \mathbb{R}$ such that $v(t) = C - u(t)$ and $u'(t) \neq 0$ for all $t \in \mathbb{R}$.

Q8 (a) Let $S \subset \mathbb{R}^3$ be a regular surface which is globally parametrised by $\mathbf{x} : \mathbb{R}^2 \to S$,

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2 \right).$$

Calculate the coefficients E, F, G of the first fundamental form with respect to $\mathbf{x}$. Verify that $\mathbf{x}$ is isothermal.

(b) Show that the coefficients L, M, N of the second fundamental form with respect to $\mathbf{x}$ in (a) are given by $L = 2, M = 0, N = -2$. You can use without proof that

$$\mathbf{N}(\mathbf{x}(u,v)) = \frac{1}{1 + u^2 + v^2}(-2u, 2v, 1 - u^2 - v^2)$$

is a Gauss map of S.

(c) Calculate the principal curvatures of the surface S and show that the Gauss curvature of S is given by

$$K(\mathbf{x}(u,v)) = -\frac{4}{(1 + u^2 + v^2)^4}.$$