EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3031-WE01

Title:
Number Theory III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Let K be a number field and write $R = \mathcal{O}_K$ for its ring of integers.

(a) Let I be a non-zero proper ideal in R. Show that $I \cap \mathbb{Z}$ is a non-zero proper ideal of $\mathbb{Z}$.

(b) Show that there exist infinitely many prime ideals in R. (Here you can use without proof the fact that there exist infinitely many primes in $\mathbb{Z}$).

Q2 (a) Let $\alpha := \frac{1}{3^{4/3}} - \frac{1}{3^{2/3}}$, where $\frac{1}{3^{4/3}}, \frac{1}{3^{2/3}} \in \mathbb{R}$. Show that α is an algebraic number. Further, give an $m \in \mathbb{Z}$ such that $m\alpha \in \overline{\mathbb{Z}}$.

(b) Write $\beta := (5+\sqrt{17})^{1/3} + (5-\sqrt{17})^{1/3}$, where $(5+\sqrt{17})^{1/3}, (5-\sqrt{17})^{1/3} \in \mathbb{R}$. Show that β is an algebraic integer and find the degree $[\mathbb{Q}(\beta) : \mathbb{Q}]$. Justify your answer.

Q3 (a) Find the fundamental unit of $\mathbb{Q}(\sqrt{10})$.

(b) Find two distinct solutions $(x, y) \in \mathbb{N} \times \mathbb{N}$, $x, y > 1$, to the equation

$$x^2 - 10y^2 = 1.$$

Q4 Let $K = \mathbb{Q}(\sqrt{-29})$ and $R = \mathcal{O}_K = \mathbb{Z}[\sqrt{-29}]$.

(a) Find all the prime ideals of R of norm 3 or of norm 11, respectively.

(b) Find an element $\alpha \in R$ of norm 33 and use this to show that there is a prime ideal $\mathfrak{p}$ of R of norm 3 and a prime ideal $\mathfrak{q}$ of R of norm 11 such that

$$[\mathfrak{p}] = [\mathfrak{q}]$$

in the class group of R.

SECTION B

Q5 Let p be an odd prime and set $\zeta := e^{\frac{2\pi i}{p}} \in \mathbb{C}$. We set $K := \mathbb{Q}(\zeta)$.

(a) Show that the minimal polynomial of ζ over $\mathbb{Q}$ is $\Phi(x) = x^{p-1} + x^{p-2} + \ldots + x + 1$.

(b) Show that $p = u(1-\zeta)^{p-1}$ for some $u \in \mathcal{O}_K^\times$.

(c) Show that $1 - \zeta$ is an irreducible element in $\mathcal{O}_K$.

(d) Is there a field extension F of K with $[F : K]$ finite, such that the element $1 - \zeta$ is not an irreducible element as an element in $\mathcal{O}_F$? Justify your answer.

Q6 (a) Let p and q be two distinct odd primes. Given that there exists at least one solution, find how many solutions there are to the equation

$$a^2 + 2b^2 = p^7 q^9,$$

with a and b in $\mathbb{Z}$. Here you may use without proof the fact that the ring $\mathbb{Z}[\sqrt{-2}]$ is a U.F.D.

(b) Let $K = \mathbb{Q}(\sqrt{-29})$ and write $R = \mathcal{O}_K$ for its ring of integers. Show that there exist non-principal ideals of norm 33 in R. Find one of them by giving generators for it.

Q7 (a) Let $K = \mathbb{Q}(\theta)$, where $\theta^3 - \theta - 3 = 0$. Compute the discriminant of $\mathbb{Z}[\theta]$.

(b) Let $K = \mathbb{Q}(\sqrt{7})$ and $R = \mathcal{O}_K = \mathbb{Z}[\sqrt{7}]$. Find a generating basis and compute the discriminant of the ideal

$$I = (2 + \sqrt{7})_R.$$

Q8 Let $K = \mathbb{Q}(\sqrt{-91})$ and $R := \mathcal{O}_{-91} = \mathbb{Z}[\frac{1+\sqrt{-91}}{2}]$.

(a) Find the class number of K. You may use the Minkowski bound, given by $B_K = \left(\frac{4}{\pi}\right)^t \frac{n!}{n^n}\sqrt{|\Delta_K|}$.

(b) Let $\alpha = x + \sqrt{-91} \in R$. Find three prime ideals P_1, P_2, P_3 of R such that any prime ideal of R that divides both (α) and $(\overline{\alpha})$ must be one of the three prime ideals P_1, P_2 or P_3.

(c) Find all the solutions $(x, y) \in \mathbb{Z}^2$ (if any) to the equation

$$x^2 + 91 = y^3.$$

(Hint: Use the previous parts.)