Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2022 | MATH3041-WE01 |

**Title:** Galois Theory III

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| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

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| **Revision:** | |

ED01/2022

## SECTION A

**Q1** **1.1** List all irreducible quadratic monic polynomials in $\mathbb{F}_3[T]$.

**1.2** How many irreducible cubic monic polynomials are there in $\mathbb{F}_3[T]$? (Justify your answer.)

**Q2** **2.1** Find the degree $[\mathbb{Q}(\sqrt[5]{3}, \sqrt[7]{5}) : \mathbb{Q}]$.

**2.2** Prove that the polynomial $T^7 - 5$ is irreducible in the ring of polynomials $\mathbb{Q}(\sqrt[5]{3})[T]$.

**Q3** Which of the following field extensions $L/K$ are Galois? (Justify your answer.):

**3.1** $K = \mathbb{Q}(i)$, $L = K(\sqrt[4]{7}, \sqrt[4]{11})$;

**3.2** $K = \mathbb{R}(X^3)$, $L = \mathbb{R}(X)$;

**3.3** $K = \mathbb{F}_7(X^3)$, $L = \mathbb{F}_7(X)$;

**3.4** $K = \mathbb{F}_3(X^3)$, $L = \mathbb{F}_3(X)$.

**Q4** **4.1** Suppose $K$ is a finite field of characteristic $p$. Prove that the map $\sigma : K \to K$ such that for any $a \in K$, $\sigma(a) = a^p$, is an automorphism of the field $K$.

**4.2** Let $k = \mathbb{F}_2(\alpha)$, where $\alpha$ is a root of $g(T) = T^4 + T + 1 \in \mathbb{F}_2[T]$. Find all irreducible factors of $g(T)$ in the ring of polynomials $k[T]$.

## SECTION B

**Q5** **5.1** (a) Suppose the characteristic of a field $K$ does not equal 2 and $L/K$ is a field extension of degree 2. Prove that there is an $a \in K$ such that $L = K(\sqrt{a})$.

(b) Give an example of a quadratic field extension $F/E$ such that $F$ does not appear in the form $E(\sqrt{b})$, where $b \in E$.

**5.2** Find the Galois group of the polynomial $F(X) = X^3 - 2t^2X - t^3 \in K[X]$, where

(a) $K = \mathbb{Q}(t)$;

(b) $K = \mathbb{F}_5(t)$.

(In both cases $K$ is the field of rational functions in the variable $t$ with coefficients in $\mathbb{Q}$ and $\mathbb{F}_5$, respectively.)

**Q6** For any $n \in \mathbb{N}$, let $\zeta_n \in \mathbb{C}$ be an $n$-th primitive root of unity.

**6.1** (a) Prove that $\{\zeta_7^m \mid m = 1, 2, 3, 4, 5, 6\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}(\zeta_7)$.

(b) Find the rational numbers $a_i$, where $1 \leqslant i \leqslant 6$, such that

$$\sqrt{-7} = a_1\zeta_7 + a_2\zeta_7^2 + a_3\zeta_7^3 + a_4\zeta_7^4 + a_5\zeta_7^5 + a_6\zeta_7^6 .$$

**6.2** Let $L = \mathbb{Q}(\zeta_{28})$. Prove that $\sqrt{7} \in L$ and find the structure of the group $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{7}))$.

**6.3** Let $M = \mathbb{Q}(\zeta_{288})$. How many subfields $N$ such that $[N : \mathbb{Q}] = 2$ are there in $M$? List all these fields $N$ in the form $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$.

**6.4** Find the degree $[\mathbb{Q}(\zeta_{19} + \zeta_{19}^7 + \zeta_{19}^{11}) : \mathbb{Q}]$.

**Q7** Let $K = \mathbb{Q}(T)$ be the field of rational functions in the variable $T$ with coefficients in $\mathbb{Q}$. Let $L$ be a minimal normal over $K$ field extension of $K(\sqrt[4]{T})$.

**7.1** Find the field $L$ and explain why $L$ is Galois over $K$. Find generators and relations for the group $\mathrm{Gal}(L/K)$.

**7.2** List all subgroups $H$ of order 4 in $G$ and find the corresponding subfields $L^H$ in $L$.

**7.3** List all subgroups $\widetilde{H}$ of order 2 in $G$ and find the corresponding subfields $L^{\widetilde{H}}$ in $L$.

**Q8** Let $K = \mathbb{Q}(\Theta)$, where $\Theta = \sqrt{1 + \sqrt{-5}}$.

**8.1** Find a minimal Galois extension $L$ of $\mathbb{Q}$ containing $K$.

**8.2** Find generators and relations for the group $\mathrm{Gal}(L/\mathbb{Q})$.

**8.3** Prove that $\sqrt{-30} \in L$ and find $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{-30}))$.

**8.4** Prove the existence of $A \in \mathbb{Q}(i, \sqrt{30})$ such that $L(i) = \mathbb{Q}(i)(\sqrt[4]{A})$. Find $A$.