Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3071-WE01

Title:
Decision Theory III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 A particular coin, when spun, may have probability $p = 0.8$ of landing heads or probability $p = 0.2$ of landing heads. Your prior probability that $p = 0.8$ is 0.6. You must guess whether $p = 0.8$ or $p = 0.2$ with reward 10 utility units if correct, zero if wrong.

(a) Before making your choice, you are allowed to spin the coin once and observe the outcome. This observation costs c utility units, which is only paid if your guess is successful, so that your winnings, given a correct guess, are $10 - c$, while guessing wrong still rewards zero. Find the largest value of c for which you would choose to observe the coin spin.

(b) Suppose that you may choose to observe a spin of the coin, and, having seen the outcome of this spin, then you may make an immediate decision or you may choose to observe a second spin of the coin, at which point you must make a decision. Each observation of a coin spin costs c, if your guess is successful, so that, if you observe two coin spins, your winnings given a correct guess are $10 - 2c$, while guessing wrong still rewards zero. Find the largest value of c for which you would choose, for at least one possible outcome of the first spin, to observe the second coin spin.

Q2 (a) In a particular decision problem, the rewards depend on two attributes, W and T. Explain what it means to say that, for you, attributes W and T are mutually utility independent. State the form of the utility function if W and T are mutually utility independent.

(b) Suppose that you are choosing between jobs and the two attributes that you consider most important are the wages, W, and the amount of time worked, T. You view W and T as mutually utility independent.

Suppose that your marginal utilities for W and T within the range of values of interest are linear, i.e. that $U(W) = W,\ U(T) = -T$.

Suppose that you are indifferent between

[**A:1**] a job with $T = 100, W = 1,500$

[**A:2**] a job with $T = 250, W = 3,500$

and that you are also indifferent between

[**B:1**] a job with $T = 200, W = 2,000$

[**B:2**] a job with $T = 150, W = 1,500$

Assigning utility zero to the pair $T = 250, W = 1,500$, evaluate your utility function for jobs as a function of T and W. Interpret what the utility function reveals about your attitude towards the two attributes.

Q3 Five people, denoted A–E, have individual preference orderings over five options, denoted a–e, as given in the table below, where for simplicity of notation preference relations of individuals are denoted without the individual being indicated by a subscript, so e.g. $a < b$ for person A means $a <_A b$.

A	$a < b < c < d < e$
B	$d \sim b < e < c < a$
C	$e \sim d \sim c < b < a$
D	$a < b < e < d < c$
E	$c < a \sim b \sim e < d$

They agree to use the following general procedure for combining their individual preferences into a group preference ordering:

Suppose that there are k options. For each person i, the options get scores $s_i(\cdot)$, which in case of strict preferences only are integers from 1 to k, with 1 for the least preferred option and k for the most preferred option. In case of indifference among two or more options, those options get the corresponding average score. For example, for person B the scores are $s_B(d) = s_B(b) = 1.5, s_B(e) = 3, s_B(c) = 4, s_B(a) = 5$. Next, for each option x with set of individual scores $\{s_i(x), i = A, \ldots, E\}$, the median value of this set is defined as the group score $S(x)$. Finally, the group preference ordering is defined based on these group scores in the logical manner, with higher score reflecting higher group preference, so e.g. $x >_g y$ if and only if $S(x) > S(y)$ and $x \sim_g y$ if and only if $S(x) = S(y)$.

(a) Apply this procedure to the preference orderings in the table above, to derive the group preference ordering.

(b) State the axioms in Arrow's Impossibility Theorem, and for each axiom explain in detail whether or not it is satisfied by the procedure above. For each axiom which is not satisfied, illustrate this using the preference orderings given above.

(c) Suppose that the final step in the procedure above is changed so that higher group preference is defined by lower group score, so e.g. $x >_g y$ if and only if $S(x) < S(y)$, while still $x \sim_g y$ if and only if $S(x) = S(y)$. Explain in detail whether or not this change to the procedure leads to a change of the axioms being satisfied, when compared to part (b).

Q4 Consider the Traveller's Dilemma game, where you and another traveller are asked, independently, to write down the price of a similar broken item, for compensation; it must be an integer from 2 to 100. If both of you write down the same number, you both receive that amount. If the numbers differ, both of you receive the lower amount, with an extra reward of 2 for the person who reported the lower number and a penalty of 2 for the person who reported the higher number.

(a) State the general definition of a Nash equilibrium for games with two players. Specify the Nash equilibrium for this game and prove that it is the unique Nash equilibrium.

(b) Suppose that, before you write down your number, you learn that the other traveller wrote down either the number 90 or 95, and you assign probability 0.5 to each of these. Derive your optimal strategy assuming you want to maximise your expected pay-off. Also explain what your optimal strategy would be if, instead, you would aim at maximising your minimal pay-off.

SECTION B

Q5 (a) Explain what is meant by a utility function on a set of rewards. Explain the relevance of utility functions to decision making under uncertainty.

(b) Prove that a utility function on a set of rewards, for an individual, is unique up to a positive linear transform. State clearly any assumptions on the preferences of the individual which are required for this proof.

(c) Suppose that the utility function of an individual for positive amounts of money, X, is of the form $U(X) = \log(aX + b)$, where a, b are positive constants. Explain why, in certain circumstances, this function might be a better choice than $V(X) = X$ to express utility for the individual.

Discuss carefully how changes in the values of a and b relate to changes in the attitude to risk represented by $U(X)$.

Q6 We wish to estimate the parameter, λ, of a Poisson distribution. Our prior probability distribution for λ is a gamma distribution, with parameters $\alpha, \beta > 1$, and loss function, for estimate d and value λ, of form

$$L(\lambda, d) = \frac{(\lambda - d)^2}{\lambda}$$

(a) Find the Bayes rule and the Bayes risk for an immediate decision.

(b) Suppose that we take an independent sample $X_1, \ldots, X_n$, of size n from the Poisson distribution before choosing the decision.

Derive the posterior distribution for λ for a given sample.

Hence, find the Bayes rule and Bayes risk after sampling.

(c) Find the Bayes risk of the sampling procedure, for given n.

(d) Suppose that the cost of the sample in the above problem depends on the observed sample outcomes as follows. If a sample of size n is taken, with observed values $X_1 = x_1, \ldots, X_n = x_n$, then the cost is $c(x_1^2 + \ldots + x_n^2)$, where c is a positive constant. Find the optimal choice of sample size.

The Poisson distribution, parameter λ, has probability function

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

The gamma distribution, parameters $\alpha, \beta > 0$ has probability density function

$$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}, \quad y > 0$$

Q7 Two people, U and V, agree to solve a joint decision problem via bargaining. They have four options, A, B, C, D, for which their individual utilities are as follows

	A	B	C	D
U	1	5	8	8
V	8	6	3	0

They decide that, if they fail to reach agreement, they will settle for a fifth option, for which they both have utility 1.

(a) Identify the Pareto boundary and the status quo point for this problem. Find the Nash point for this problem by computation, using its definition. Specify the bargain corresponding to the Nash point.

(b) Derive the Nash point by a geometric method and explain which Nash axioms are used at each step (it is enough to name the axioms, you do not need to specify them further).

(c) Derive the equitable distribution point for this bargaining problem and specify the corresponding bargain.

(d) The Nash point and equitable distribution points are each unique solutions to a set of six axioms, five of which are the same. Write down the axioms which are different for these points, so one for each, and explain briefly how they differ.

(e) Suppose that a further option, E, becomes available, for which U has utility 5 and V has utility 6. Explain whether or not this makes any difference to the Nash point, the equitable distribution point, and their corresponding bargains.

Q8 Consider the following pay-off table for a two-person zero-sum game, where R chooses $R1$ or $R2$, and C chooses $C1, C2, C3$ or $C4$. The pay-offs to R are as follows

	$C1$	$C2$	$C3$	$C4$
$R1$	1	6	9	4
$R2$	7	1	3	4

The pay-off to C is minus the pay-off to R.

(a) Find the minimax strategies for R and C and the value of this game.

(b) Consider the same game with the pay-offs specified in the table above. Suppose now that, instead of playing minimax strategies, each player determines their strategy as follows, independently of the other player. R simply tosses a fair coin, so chooses either $R1$ or $R2$, each with probability $1/2$. C aims to maximize his expected pay-off, where he assumes that R will play $R2$ with probability $p \in [0, 1]$ and $R1$ with probability $1 - p$. Determine the resulting expected pay-off to C of this game, for each value of $p \in [0, 1]$.

(c) Suppose that C finds it instinctively attractive to opt for $C4$, and after reflecting on this decides to quantify his preferences using utilities for the outcomes. Explain whether or not the solution method of part (a) could still be used. If it can be used, give a detailed justification with explanation of any further assumptions; if it cannot be used, explain any complications in solving the game.